Sanya, China

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Lectures on AdS/CFT Quantum Spectral Curve and Fishnet CFT

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Plan

Lecture I

- Many faces of N=4 SYM: AdS/CFT, quantum and classical planar integrability, Fishnet CFT, BPS physics and localization, black holes, etc
- Field content, classification of operators and symmetries of N=4 SYM

Lecture II

- 1-loop dilatation operator as a Heisenberg (super)spin chain Hamiltonian.
- Integrability and spectrum via algebraic Bethe ansatz for SU(2) sector
- T- and Q-systems and (Super)Hasse diagram for SU(N) case

Lecture III

- All-loop integrability for spectrum: AdS/CFT quantum spectral curve (QSC).
- An application: PT for conformal dimensions of twist-2 operators
- Review of results (QSC numerics and PT, BFKL, cusp) and perspectives (structure constants, correlators, BH)

Lecture IV

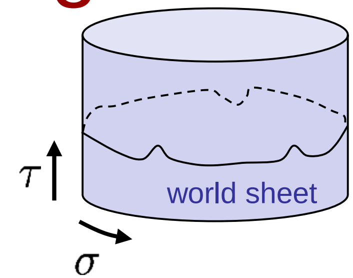
- From gamma-deformed N=4 SYM to Fishnet CFT (FCFT)
- Integrability of FCFT. More general FCFTs (Loom for FCFTs)
- Applications: multi-loop graphs from QSC, BFKL…

Lecture I

Fields, operators and symmetries of N=4 Super-Yang Mills theory

N=4 SYM dual to superstring on $AdS_5 \times S^5$

$$\mathcal{S}_{SYM} = \frac{1}{\lambda}\int d^4x \, \mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + \bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\Phi\Psi + [\Phi,\Phi]^2\right)$$



$$\lambda = g_{YM}^2 N = \frac{R^4}{\alpha'^2} = g_s N$$

super-conformal theory:
β-function=0, no massive particles

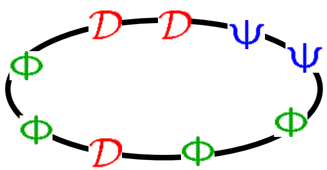
$$\mathcal{S}_{sigma} = \sqrt{\lambda}\int d\tau \int_0^L d\sigma \, \left[\left(\partial \vec{X}\right)^2 + \left(\partial \vec{Y}\right)^2 + \text{fermions}\right]$$

Maldacena
Gubser,Klebanov,Polyakov
Witten

Metsaev, Tseytlin

Exact equivalence

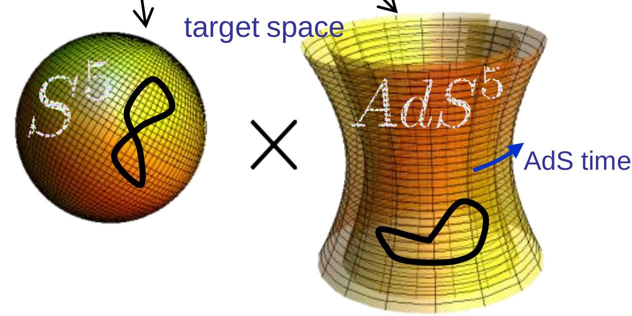
$$\mathcal{O}(x) = \mathrm{Tr}\left[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots\right](x)$$



CFT/AdS duality

$\longleftrightarrow$

weak / strong



$$\mathcal{O}_A(\xi x) \rightarrow \xi^{\Delta_A(\lambda)}\mathcal{O}_A(x)$$

$$\hat{D}\,\mathcal{O} = \Delta\,\mathcal{O}$$

$$AdS_5: \quad -X_{-1}^2 - X_0^2 + X_7^2 + X_8^2 + X_9^2 + X_{10}^2 = -R^2$$

$$S^5: \quad X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = R^2$$

Anomalous dimension $\Delta_{\mathcal{O}}$ = Energy of the dual string state

Super-conformal symmetry PSU(2,2|4) isometry of string target space
($AdS_5 \times S^5$ is its bosonic part).

Gromov, V.K., Leurent, Volin '13, '14

Quantum Spectral Curve – solution at any coupling

Conformal transformations SO(2,4)~SU(2,2)

- Scale invariance: stress tensor is traceless, which implies a larger symmetry
- Conformal invariance: local scale invariance

$$x^\mu \to x'^\mu, \qquad dx'^2 = \rho^2(x) dx^2$$

Infinitesimal: $$\delta x^\mu = v^\mu(x), \qquad \partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d}\eta_{\mu\nu}\,\partial\cdot v\,,$$

$$v^\mu = \left\{a^\mu, \omega^\mu{}_\nu\, x^\nu, x^\mu, x^2 a^\mu - 2(a\cdot x)x^\mu\right\}$$

(translation P, Lorentz L, dilatation D, special conf. K)

K := Spec.conf. = inversion × shift × inversion

$$\rho(x) = 1 + \frac{\partial\cdot v}{d}$$

Isometry: $\rho(x) = 1$

- Full conformal group:

$$\begin{aligned} D &= -ix_\mu \partial_{x_\mu} - i\Delta \\ L_{\mu\nu} &= ix_\mu \partial_{x_\nu} - ix_\nu \partial_{x_\mu} \\ P_\mu &= -i\partial_{x_\mu} \\ K_\mu &= 2\,x^\nu\, L_{\nu\mu} - ix^2 \partial_{x_\mu} - 2i\Delta x_\mu \end{aligned}$$

$$\begin{aligned} [L_{\mu\nu}, L_{\lambda\rho}] &= \eta_{\mu\lambda}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\lambda} - \eta_{\mu\rho}L_{\nu\lambda} - \eta_{\nu\lambda}L_{\mu\rho} \\ [L_{\mu\nu}, P_\lambda] &= \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu \\ [L_{\mu\nu}, K_\lambda] &= \eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu \\ [D, P_\mu] &= P_\mu \\ [D, K_\mu] &= -K_\mu \\ [P_\mu, K_\nu] &= 2L_{\mu\nu} - 2\eta_{\mu\nu}D. \end{aligned}$$

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Conformal 2- and 3-point function

- We consider only planar limit. Operators from local fields

$$\mathcal{O}(x) = \text{linear combinations of } \mathrm{Tr}\left[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots\right](x)$$

- 2- and 3-point correlators (structure functions):

$$G_{II}(x_1,x_2) = \langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle \qquad G_{III}(x_1,x_2,x_3) = \langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}(x_3)\rangle$$

- Under special conformal: $\delta \log(x_1-x_2)^2 = -a\cdot(x_1+x_2)\,, \qquad \rho(x) = 1-2a\cdot x+O(a^2)$

$$\delta G_{II}(x_1,x_2) = -\frac{\partial G_{II}}{\partial \log r_{12}} a\cdot(x_1+x_2) = 2(\Delta_1 a\cdot x_1+\Delta_2 a\cdot x_2)G_{II}$$

- Possible only for equal dimensions, then $\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{\delta_{ij}}{|x|^{2\Delta_j(\lambda)}}$

- Similarly, for 3-point corr.:

$$\delta G_{III}(x_1,x_2,x_2) = -\sum_{i>j}\frac{\partial G_{III}}{\partial \log r_{ij}} a\cdot(x_i+x_j) = 2\sum_i \Delta_i a\cdot x_i G_{III}$$

- Solution: $\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}(\lambda)}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}|x_{31}|^{\Delta_i+\Delta_k-\Delta_j}}$

They describe the whole conformal theory via operator product expansion

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k \frac{C_{ijk}(\lambda)}{|x|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(0) + \text{decendants}$$

*

Zero-magnon 4-point correlator and exact OPE data

- Exact expression of a 4-point correlator (only from conformal symmetry!)

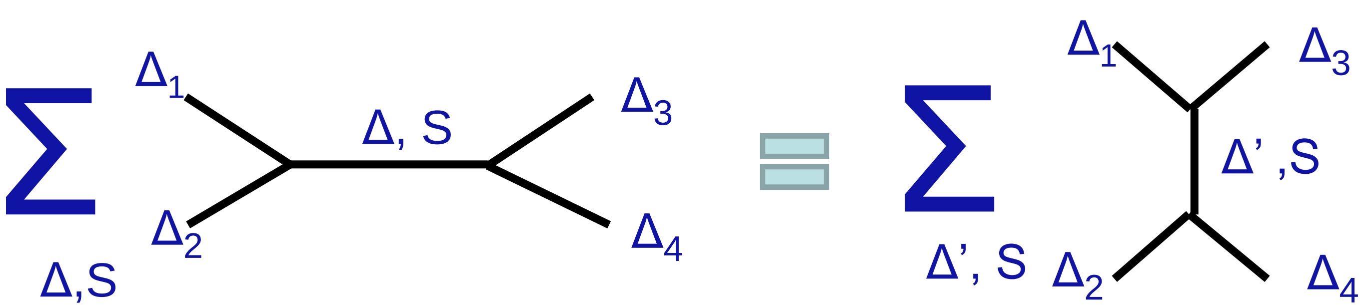
$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \mathcal{G}(u,v)\frac{(x_{24}/x_{14})^{\Delta_1-\Delta_2}(x_{14}/x_{13})^{\Delta_3-\Delta_4}}{x_{12}^{\Delta_1+\Delta_2}x_{34}^{\Delta_3+\Delta_4}}$$

$$\mathcal{G}(u,v) = \sum_{\Delta,S} C_{12}^{S,\Delta}\, C_{34}^{S,\Delta}\, u^{\frac{\Delta-S}{2}} g_{\Delta,S}\left(\frac{\Delta-\Delta_1+\Delta_2-S}{2}, \frac{\Delta-\Delta_4+\Delta_3-S}{2}; u,v\right)$$

cross-ratios

$$u = z\bar{z} = x_{12}^2 x_{34}^2/(x_{13}^2 x_{24}^2)$$
$$v = (1-z)(1-\bar{z}) = x_{14}^2 x_{23}^2/(x_{13}^2 x_{24}^2)$$

- Conformal block g(…) explicitly expressed through hypergeometric function ${}_2F_1(a,b,c;z)$, where a,b,c are linear functions of spin S and dim Δ
- OPE for 4-point correlator in two different channels:



crossing

Gliozzi,Scherk,Olive'77

N=4 SYM as a superconformal 4d QFT

- Can be realized as reduction to 4D of 10D N=1 SYM (so(6) setting)

$$\mathcal{L}_{\rm YM}[\mathcal{W}] = \tfrac{1}{4}\,{\rm Tr}\,\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \tfrac{1}{2}\,{\rm Tr}\,\mathcal{D}^{\mu}\Phi^{n}\mathcal{D}_{\mu}\Phi_{n} - \tfrac{1}{4}g^{2}\,{\rm Tr}\,[\Phi^{m},\Phi^{n}][\Phi_{m},\Phi_{n}] \qquad ($$

$$+\,{\rm Tr}\,\dot{\Psi}^{a}_{\dot\alpha}\sigma^{\dot\alpha\beta}_{\mu}\mathcal{D}^{\mu}\Psi_{\beta a} - \tfrac{1}{2}ig\,{\rm Tr}\,\Psi_{\alpha a}\sigma^{ab}_{m}\varepsilon^{\alpha\beta}[\Phi^{m},\Psi_{\beta b}] - \tfrac{1}{2}ig\,{\rm Tr}\,\dot{\Psi}^{a}_{\dot\alpha}\sigma^{m}_{ab}\varepsilon^{\dot\alpha\dot\beta}[\Phi_{m},\dot{\Psi}^{b}_{\dot\beta}]$$

- Fields in adjoint irrep of gauge group su(N_c) - $N_c \times N_c$ matrices
- Fields - tensors with global superconformal $psu(2,2|4)$

Gauge field
Lorentz symmetry SO(4)

$$\mathcal{D}_\mu = \partial_\mu + iA_\mu$$

$$X = \Phi_1 + i\Phi_4$$

$$Y = \Phi_2 + i\Phi_5$$

$$Z = \Phi_3 + i\Phi_6$$

Scalar fields
R-symmetry SO(6)~SU(4)

$$\Phi_m\,, \qquad m = 1,2,\ldots,6$$

16 Weyl spinors: $\Psi_{\alpha a},\ \Psi^{\dot\alpha}_{a},\quad a = 1,\ldots,4 : su(4),\quad \alpha,\dot\alpha = 1,2 : so(4) \simeq su(2)_L \times su(2)_R$

Gamma matrices for 4D spinors and 6D spinors

$$\sigma^{\{\mu}\sigma^{\nu\}} = \eta^{\mu\nu}, \qquad \sigma^{\{m}\sigma^{n\}} = \eta^{mn}$$

$$\sigma^{m,ab} = \tfrac{1}{2}\varepsilon^{abcd}\sigma^{m}_{cd}, \qquad \sigma_{m,ab} = \tfrac{1}{2}\varepsilon_{abcd}\sigma^{cd}_{m}$$

Fierz id. 4D: $\sigma^{\dot\alpha\beta}_{\mu}\sigma^{\mu}_{\dot\gamma\delta} = 2\delta^{\dot\alpha}_{\dot\gamma}\delta^{\beta}_{\delta}\,, \qquad \sigma^{\dot\alpha\beta}_{\mu}\sigma^{\mu,\dot\gamma\delta} = 2\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\beta\delta}, \qquad \sigma_{\mu,\dot\alpha\beta}\sigma^{\mu}_{\dot\gamma\delta} = 2\varepsilon_{\dot\alpha\dot\gamma}\varepsilon_{\beta\delta}$

Fierz id. 6D: $\sigma^{ab}_{m}\sigma^{m}_{cd} = 2\delta^{a}_{d}\delta^{b}_{c} - 2\delta^{a}_{c}\delta^{b}_{d}, \qquad \sigma^{ab}_{m}\sigma^{m,cd} = -2\varepsilon^{abcd}, \qquad \sigma_{m,ab}\sigma^{m}_{cd} = -2\varepsilon_{abcd}$

SU(4) notations and topological term

- Spinorial notations for fields:

$$\mathcal{F}_{\pm}^{\mu\nu} = \pm\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}\mathcal{F}_{\pm\sigma\rho}$$

$$\sigma^{\mu\nu}_{\alpha\beta} = \sigma^{[\mu}_{\alpha\dot\alpha}\sigma^{\nu]}_{\beta\dot\beta}\,\epsilon^{\dot\alpha\dot\beta}$$

$$\mathcal{F}_{+\alpha\beta} \equiv \frac{1}{2}\sigma^{\mu\nu}_{\alpha\beta}\mathcal{F}_{+\mu\nu} = \mathcal{F}_{+\beta\alpha}, \quad \mathcal{F}_{-\dot\alpha\dot\beta} \equiv \frac{1}{2}\sigma^{\mu\nu}_{\dot\alpha\dot\beta}\mathcal{F}_{-\mu\nu} = \mathcal{F}_{-\dot\beta\dot\alpha} \qquad \mathcal{D}_{\alpha\dot\beta} \equiv \sigma^{\mu}_{\alpha\dot\beta}\mathcal{D}_\mu$$

$$\Phi_{ab} = \Phi_m\,\sigma^m_{ab}. \qquad \Phi_m, \quad m = 1,\dots,6 \in SO(6); \quad \Phi^{ab} \in SU(4) \sim SO(6)$$

$$\Phi_{ab} = \begin{pmatrix} 0 & \Phi_3 + i\Phi_6 & -\Phi_2 - i\Phi_5 & \Phi_1 - i\Phi_4 \\ -\Phi_3 - i\Phi_6 & 0 & \Phi_1 + i\Phi_4 & \Phi_2 - i\Phi_5 \\ \Phi_2 + i\Phi_5 & -\Phi_1 + i\Phi_4 & 0 & \Phi_3 - i\Phi_6 \\ -\Phi_1 + i\Phi_4 & -\Phi_2 + i\Phi_5 & -\Phi_3 + i\Phi_6 & 0 \end{pmatrix}$$

- Lagrangian in these notations:

$$\mathcal{L} = N_c \mathrm{tr}\Big[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^{\alpha}_{\dot\alpha}\Phi^{\dagger}_{ab}D^{\dot\alpha}_{\alpha}\Phi^{ba} + \frac{g^2}{8}[\Phi^{\dagger}_{ab},\Phi^{\dagger}_{cd}][\Phi^{ba},\Phi^{dc}] + \\ + 2i\bar\psi^{\dot\alpha}_{a}D^{\alpha}_{\dot\alpha}\psi^{a}_{\alpha} - \sqrt{2}g\psi^{\alpha\, a}[\Phi^{\dagger}_{ab},\psi^{b}_{\alpha}] + \sqrt{2}g\bar\psi_{\dot\alpha\, a}[\Phi^{ab},\bar\psi^{\dot\alpha}_{b}]\ \Big]$$

- Topological term:

$$\theta\int d^4x\,\mathrm{Tr}F\wedge F$$

complexified coupling

$$\tau_{YM} = \frac{1}{g^2_{YM}} + \frac{\theta}{2\pi} \Leftarrow \tau_{str} = \frac{i}{g_s} + \chi \qquad g^2 = N_c g^2_{YM}$$

SL(2,Z) S -duality, related to the string coupling

$$\tau = -\frac{1}{\tau}$$

Operators, planar graphs and 1/N expansion

- Local operators: single trace of products of matrix fields (only planar limit!)

$$\mathcal{O}(x) = \mathrm{tr}\ [\chi_1(x), \chi_2(x) \ldots \chi_L(x)]\ +\ \text{perm.}$$

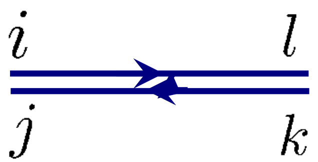
$$\chi \in \{\mathcal{D}_{\dot\alpha\beta}, \Phi_{ab}, \Psi_{\alpha b}, \Psi^{b}_{\dot\alpha}, \mathcal{F}_{\alpha\beta}, \dot{\mathcal{F}}_{\dot\alpha\dot\beta}\}$$

- Dimensions of operators can be read off from 2-point correlators

- They can be computed via perturbation theory

- Propagators in double-line notations (only color indices exposed):

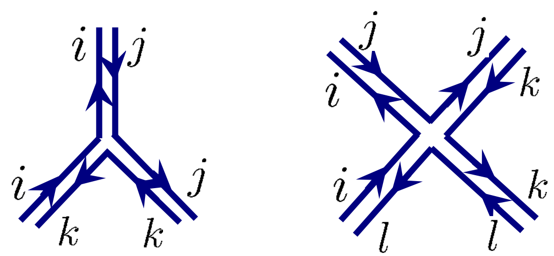
$$\left\langle \chi^{ij}(y)\chi^{kl}(x)\right\rangle_0 = \delta^{il}\delta^{jk}\, D_\chi(x-y)$$



$$i, j, k, l = 1 \ldots N_c$$

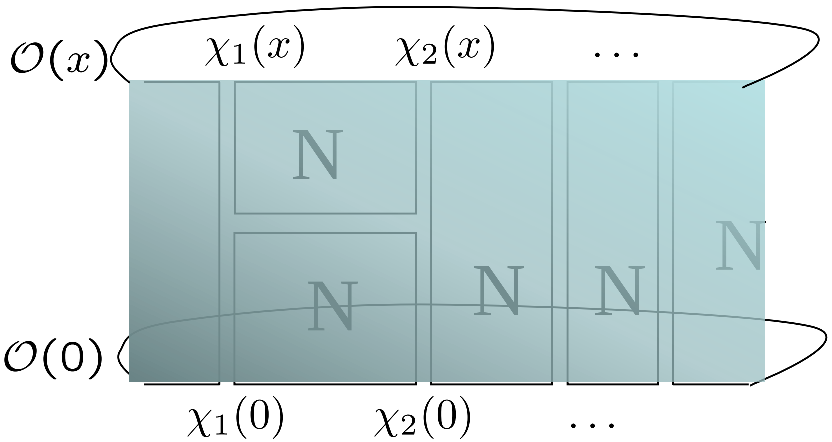
Index is conserved along each line

- Vertices

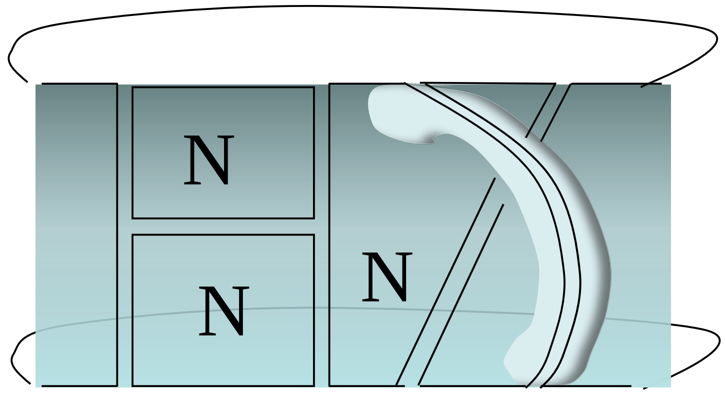


Planar graphs and large-N expansion

- Compare two cylindric graphs for two-point correlators



$(g^2_{YM}N)^5$



$(g^2_{YM}N)^5 \times \frac{1}{N^2}$

- The one with bigger number of faces (smaller genus) dominates

Double expansion: $$\langle\langle O(x)O(0)\rangle\rangle_{N,g} = \sum_{h=0}^{\infty}\frac{1}{N^{2h}}\sum_{m=0}^{\infty} g^{2m}K_m^{(h)}(|x|)$$

Perturbative, in 't Hooft coupling: $$g^2 = g^2_{YM}N$$

another frequent notation $\lambda \equiv 16\pi^2 g^2$

Topological, in string coupling: $$g_s = \frac{1}{N}$$

- We will always deal with planar quantities (in 't Hooft limit), like $K_m^{(0)}$

Superconformal Symmetry of N=4 SYM

Action obeys superconformal psu(2,2|4) symmetry.
It persists on QM level

Generators of global superconformal psu(2,2|4) symmetry:

$$\begin{array}{cccc} & \text{Lorentz} & \text{susy} & \text{translations} \\ \text{superconf.} & \multicolumn{3}{c}{\left(\begin{array}{c|c|c} L^{\alpha}_{\beta} & \dot{Q}^{a}_{\dot{\alpha}} & P_{\dot{\alpha}\beta} \\ \hline \dot{S}_{\dot{\alpha}a} & {}^{\text{su}(4)_R}\,R^{a}_{b} & Q_{\alpha a}\,{}^{\text{susy}} \\ \hline K_{\alpha\dot{\beta}} & S^{a}_{\alpha} & \dot{L}^{\dot{\alpha}}_{\dot{\beta}} \end{array}\right)} \\ & \text{spec. conf.} & \text{superconf.} & \text{Lorentz} \end{array}$$

$$a, b = 1, \ldots, 4$$

$$\alpha, \dot{\alpha}, \beta, \dot{\beta} = 1, 2$$

Some algebra relations: $\{Q, Q\} = \{S, S\} = \{Q, \dot{S}\} = \{\dot{Q}, S\} = 0$

$$\{Q, \dot{Q}\} = P, \qquad \{S, \dot{S}\} = K$$

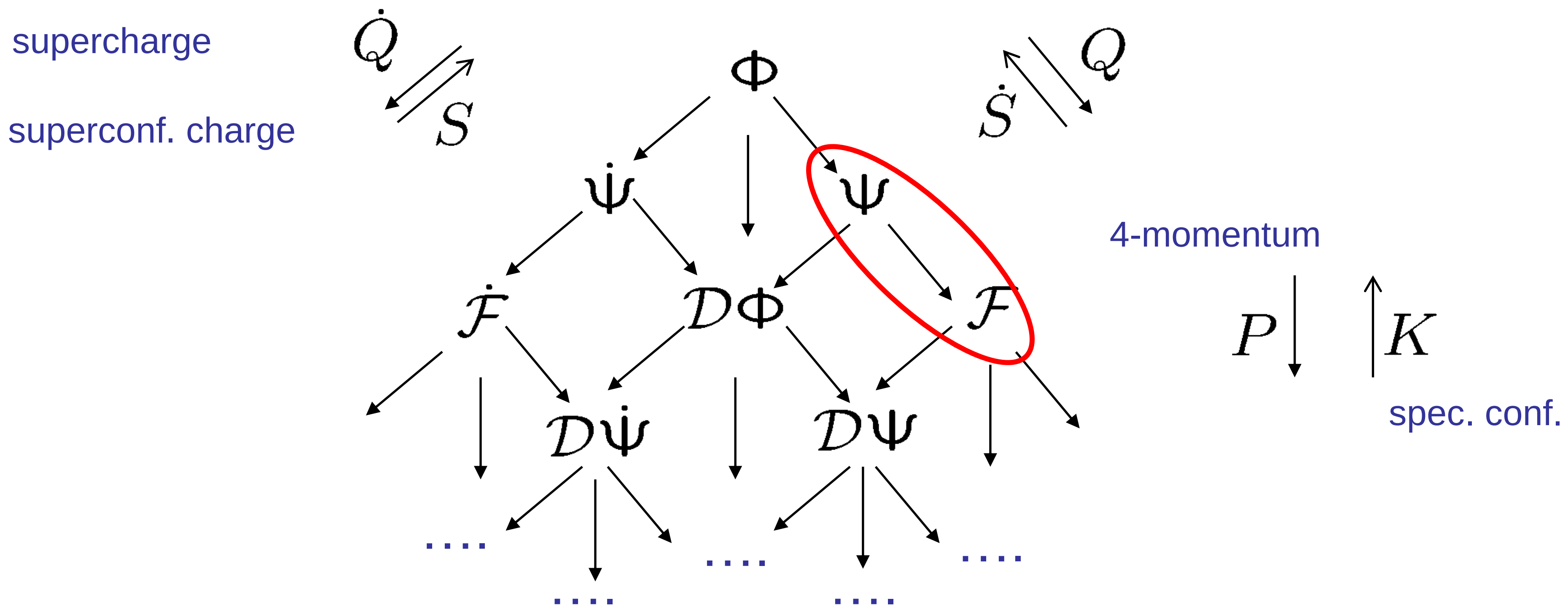
Dilatation operator is a part of algebra! $\{Q, S\} = D + R + L$

$$\{Q, K\} = S, \qquad \{D, Q\} = Q, \text{ etc}$$

Cartan charges: $\{\underset{\text{e.v. of D}}{\Delta}, \underbrace{S_1, S_2}_{\text{Lorentz}} | \underbrace{J_1, J_2, J_3}_{\text{so(6)}\sim\text{su(4)}_R}\}$

Transformation of fields in supermultiplet

SYM symmetric under global superconformal symmetry: PSU(2,2|4)



Example:

$$Q\,\Psi_{\alpha a} \;=\; -\epsilon_a^{\gamma}\mathcal{F}_{\alpha\gamma} + \frac{i}{2}\sqrt{\lambda}\varepsilon_{\alpha\beta}\epsilon_c^{\beta}\left[\Phi_{ab},\Phi^{bc}\right]$$

Operators and superalgebra

psu(2,2|4) superalgebra $\{E_{mn}, E_{kl}\} = \delta_{nk} E_{ml} - (-1)^{(p_m+p_n)(p_k+p_l)} \delta_{ml} E_{kn}$

$$m, n \in \{\underbrace{1, 2}_{\text{Lorentz}}, \underbrace{\dot{1}, \dot{2}}, \underbrace{\hat{1}, \hat{2}, \hat{3}, \hat{4}}_{su(4)_R}\}$$

$$E^{\dagger}_{mn} = (-1)^{c_m + c_n} E_{nm}$$

		α	$\dot{\alpha}$	$\hat{\alpha}$
grading	p	0	0	1
	c	0	1	0

Projectivity: $$C = \sum_n E_{nn} = 0 \Longrightarrow C = -2 - n_b + n_f + n_a = 0$$

Super-unimodularity: $$\mathrm{Str}\, E = \sum_n (-1)^F E_{nn} = 0$$

Oscillator formalism for psu(2,2|4) superalgebra (0-loop order)

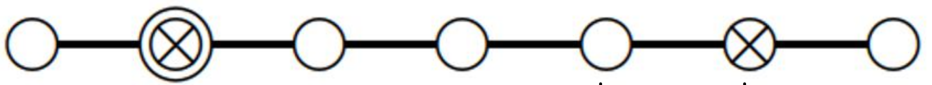
Oscillators:

$$[a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}, \quad \alpha, \beta \in \{1, 2\},$$

$$[b_{\dot\alpha}, b^\dagger_{\dot\beta}] = \delta_{\dot\alpha\dot\beta}, \quad \dot\alpha, \dot\beta \in \{\dot 1, \dot 2\},$$

$$\{f_a, f^\dagger_b\} = \delta_{ab}, \quad \hat a, \hat b \in \{\hat 1, \hat 2, \hat 3, \hat 4\}$$

Dynkin diagram



Generators (in « beauty » grading):

$$E_{mn} = \left(\begin{array}{c|c|c} -b_{\dot\alpha} b^\dagger_{\dot\beta} & -b_{\dot\alpha} f_b & -b_{\dot\alpha} a_\beta \\ \hline f^\dagger_a b^\dagger_{\dot\beta} & f^\dagger_a f_b & f^\dagger_a a_\beta \\ \hline a^\dagger_\alpha b^\dagger_{\dot\beta} & a^\dagger_\alpha f_b & a^\dagger_\alpha a_\beta \end{array}\right)$$

Bosonic generators: $E_{\alpha\beta} = a_\alpha a^\dagger_\beta, \quad E_{\dot\alpha\dot\beta} = -b_{\dot\alpha} b^\dagger_{\dot\beta}, \quad E_{\hat a\hat b} = f_{\hat a} f^\dagger_{\hat b}$

Supercharges: $E_{\dot\alpha b} = -b_{\dot\alpha} f_b, \quad \text{etc}$

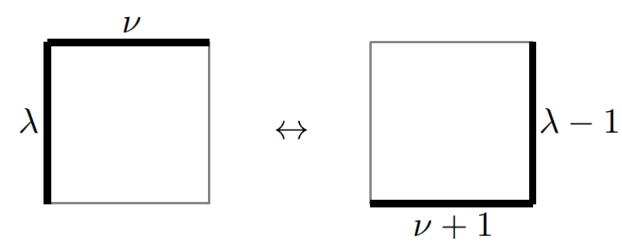
Number of fermions of type « i »: $n_f \equiv \sum_{i=1}^{4} n_{fi}$, of type « i »: $n_{fi} \equiv f^\dagger_i f_i$

Projectivity: $C = -2 - n_b + n_f + n_a = 0$

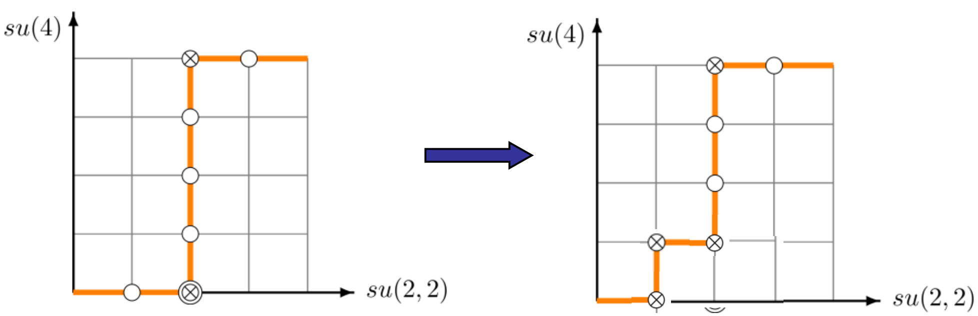
Fermionic duality and Dynkin labels

- Various gradings: Duality interchanges odd generators:

$$\begin{pmatrix} E_{mm} & E_{mn} \\ E_{nm} & E_{nn} \end{pmatrix} \leftrightarrow \begin{pmatrix} E_{nn} & E_{nm} \\ E_{mn} & E_{mm} \end{pmatrix} \qquad p_m + p_n = 1$$



e.g. $\quad E_{mn} = -b_\alpha f_a \;\rightarrow\; E_{nm} = b^\dagger_\alpha f^\dagger_a$



Name	grading	short-hand notation	Dynkin diagram
compact beauty	$12\hat{1}\hat{2}\hat{3}\hat{4}34$	2222	○—⊗—○—○—○—⊗—○
non-compact beauty	$\hat{1}\hat{2}1234\hat{3}\hat{4}$	0044	○—⊗—○—◎—○—⊗—○
compact ABA	$\hat{1}12\hat{2}\hat{3}344\hat{4}$	0224	⊗—○—⊗—○—⊗—○—⊗
non-compact ABA	$1\hat{1}\hat{2}23\hat{3}\hat{4}4$	1133	⊗—○—⊗—◎—⊗—○—⊗

- Dynkin labels:

su(4) $\qquad \lambda_a = f^\dagger_a f_a = n_{f_a}\,, \quad a = 1, \ldots, 4$

su(2,2) $\qquad \nu_i = \left\{ -b_{\dot\alpha} b^\dagger_{\dot\alpha}, a^\dagger_\alpha a_\alpha \right\}_i = \{-L - n_{b_{\dot\alpha}}, n_{a_\alpha}\}_i\,, \quad i = 1, \ldots, 4$

Fields and operators from oscillators

Bare vacuum

$$a_\alpha|0\rangle = b_{\dot\alpha}|0\rangle = f_a|0\rangle = 0$$

Hignest weight state

$$E_{mn}|\mathrm{HWS}\rangle = 0 \quad \text{for } m < n$$

Operator fixed by oscillator numbers

$$[n_{b_1}, n_{b_2} \mid n_{f_1}, n_{f_2}, n_{f_3}, n_{f_4} \mid n_{a_1}, n_{a_2}]$$

Length and bare dimension of operator (conserved only for g=0)

$$2L = n_a - n_b + n_f, \qquad \Delta_0 = \frac{n_f}{2} + n_a$$

Field interpretation		Content	Δ_0	Components
scalar	Φ_{ab}	$\mathbf{f}_a^\dagger \mathbf{f}_b^\dagger \|0\rangle$	1	6
fermion	$\Psi_{a\alpha}$	$\mathbf{f}_a^\dagger \mathbf{a}_\alpha^\dagger \|0\rangle$	$\frac{3}{2}$	8
	$\bar\Psi_{a\dot\alpha}$	$\epsilon_{abcd}\mathbf{f}_b^\dagger \mathbf{f}_c^\dagger \mathbf{f}_d^\dagger \mathbf{b}_{\dot\alpha}^\dagger \|0\rangle$	$\frac{3}{2}$	8
field strength	$\mathcal{F}_{\alpha\beta}$	$\mathbf{a}_\alpha^\dagger \mathbf{a}_\beta^\dagger \|0\rangle$	2	3
	$\bar{\mathcal{F}}_{\dot\alpha\dot\beta}$	$\mathbf{f}_1^\dagger \mathbf{f}_2^\dagger \mathbf{f}_3^\dagger \mathbf{f}_4^\dagger \mathbf{b}_{\dot\alpha}^\dagger \mathbf{b}_{\dot\beta}^\dagger \|0\rangle$	2	3
covariant derivative	$\mathcal{D}_{\alpha\dot\alpha}$	$\mathbf{a}_\alpha^\dagger \mathbf{b}_{\dot\alpha}^\dagger$	1	4

Example of operator

$$\mathrm{Tr}\,[\mathcal{Z}\mathcal{D}_{12}\Psi_{11}\mathcal{F}_{12}] = f_1^\dagger f_2^\dagger|0\rangle \otimes (a_1^\dagger)^2 b_2^\dagger f_1^\dagger|0\rangle \otimes a_1^\dagger a_2^\dagger|0\rangle$$

Super-Multiplets

Superconformal multiplet – a set of operators (irreps) related by
global psu(2,2|4) symmetry, all having the same anomalous dimension.
Contains a few conformal primaries: operators annihilated by

$$K_{\alpha,\dot\alpha} \sim a_\alpha\, a_{\dot\alpha}$$

One of them is the HWS. Which one – depends on the grading. Standart choice is the chiral primary having the lowest bare (classical) dimension

Examples

$$\mathcal{Z}\mathcal{Z}\mathcal{Y}\mathcal{Y} \rightleftarrows \mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{Y} \overset{\mathbf{f}^\dagger\mathbf{f}}{\rightleftarrows} \mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{X} \overset{\mathbf{a}^\dagger\mathbf{b}^\dagger}{\rightleftarrows} \mathcal{D}\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{X} \rightleftarrows \mathcal{D}^2\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{X} \rightleftarrows \cdots$$

$$\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{X} \;\Big\updownarrow\, \mathbf{a}^\dagger\mathbf{f} \;\; \mathcal{Z}\mathcal{Z}\mathcal{X}\Psi$$

1/2BPS multiplet (beauty grading): $\quad |\mathrm{HWS}\rangle_{\mathrm{BPS}} = \mathrm{Tr}Z^L\,, \quad n = [0,0|L,L,0,0|0,0]$

More general expression: $\quad |\mathrm{HWS}\rangle_{1/2\mathrm{BPS}} = \mathrm{Tr}(n_m\Phi^m)^L\,, \quad n_m n^m = 0$

psu(2,2|4) Cartan charges: $\quad (\Delta, S_1, S_2|J_1, J_2, J_3)$

$$\mathrm{Tr}Z^L\,, \quad (L,0,0|L,0,0)$$

$$\mathrm{Tr}X^L\,, \quad (L,0,0|0,L,0)$$

$$\mathrm{Tr}Y^L\,, \quad (L,0,0|0,0,L)$$

Conserves half of the susy. Conformal dimension at any coupling Δ=L

Dynkin labels at finite coupling

Anomalous dimension $\gamma = \Delta - \Delta_0$

Shift of Dynkin labels for conformal group

$$\nu_i = \nu_i\big|_{g=0} + \frac{\gamma}{2}\{-1,-1,1,1\}_i$$

Operators with different lengths can mix
– length is not a conserved charge at finite coupling!

$$\left.\begin{array}{l} \{L, n_{b_\alpha}\} \leftrightarrow \{L-1, n_{b_\alpha}+1\} \\ \{L, n_{f_\alpha}, n_{a_\alpha}\} \leftrightarrow \{L-1, n_{f_\alpha}-1, n_{a_\alpha}+1\} \end{array}\right\} \lambda_a \leftrightarrow \lambda_a - 1 \text{ and } \nu_i \leftrightarrow \nu_i + 1$$

Examples

n^{2222}	L	Field content example	λ_a	ν_j
$[1,1\|2,2,2,2\|1,1]$	4	$\Psi_{11}\Psi_{12}\bar{\Psi}_{11}\bar{\Psi}_{12}$	$\{2,2,2,2\}$	$\{-5,-5,1,1\}$
$[0,0\|2,2,2,2\|1,1]$	5	$\Psi_{11}\Psi_{12}\bar{\mathcal{Z}}\bar{\mathcal{X}}\bar{\mathcal{Y}}$	$\{2,2,2,2\}$	$\{-5,-5,1,1\}$
$[1,1\|3,3,3,3\|0,0]$	5	$\bar{\Psi}_{11}\bar{\Psi}_{12}\mathcal{Z}\mathcal{X}\mathcal{Y}$	$\{3,3,3,3\}$	$\{-6,-6,0,0\}$

Some conclusions to lecture I

- N=4 SYM obeys a superconformal symmetry PSU(2,2|4), uniting the Poincare and conformal bosonic subgroups
- Local operators form conformal supermultiplets, with the same anomalous dimension.
- In large N limit N=4 SYM single trace operators form a complete set
- Spectrum of conformal dimensions of these operators is encoded in the dilatation operator D(g). D is a part of superconformal algebra.
- At g=0 the generators of PSU(2,2|4) can be efficiently described via the oscillator algebra. Length L and bare (classical) Δ_0 dimension conserve
- At g≠0 the susy generators and D (and thus Δ) depend non-trivially on g.

  Mixing of operators with different lengths L.
- In the next lecture we will compute D perturbatively and diagonalize it using quantum integrability

References:

- J. Minahan, https://arxiv.org/pdf/1012.3983.pdf
- N. Beisert, https://arxiv.org/pdf/hep-th/0407277(thesis, chapt.1,2)
- C. Marboe & D. Volin, https://arxiv.org/pdf/1701.03704

Lecture II

Quantum integrability of one-loop spectrum for AdS/CFT

What is integrability of N=4 SYM?

- Integrability (in planar, large N limit) of dilation operator as a quantum Hamiltonian. « Time » = RG scaling. Today we demonstrate it at one-loop order
- Equivalence to integrable quantum spin chain for single trace operators Perturbative computations of anomalous dimensions of any operators
- Integrability is also seen on the string side -- Metsaev-Tseytlin sigma model on $AdS_5 \times S^5$ space is also integrable (both classically and QM) . Strong coupling regime available (not considered in these lectures)
- Possibility to compute the spectrum of anomalous dimensions of any single-trace operators in planar limit at any coupling. Efficient numerics for any couplings. Ultimate tool – Quantum Spectral Curve (QSC). Main goal of these lectures.
- History

1- and 2-loop integrability
Classical integrability
S-matrix, asymptotic Bethe ansatz

Bena, Roiban, Polchinski 02'
Metsaev-Tseytlin 02'
Minahan, Zarembo 03'
Beisert, Kristjansen, Staudacher 03'
V.K.,Marshakov, Minahan, Zarembo 04'
Beisert, V.K.,Sakai, Zarembo 05'

Beisert,Staudacher 04'
Beisert 05'
Janik 05'
Beisert, Eden, Staudacher '06

→ Y-system

Gromov, V.K,, Vieira '09

→ Thermodynamical Bethe ansatz (TBA)

Bombardelli,Fioravanti,Tateo '09
Gromov,V.K.,Kozak,Vieira '09
Arutyunov,Frolov '09
Cavaglia, Fioravanti, Tateo '09

→ Quantum spectral curve (QSC)

Gromov, V.K., Leurent, Volin '13, '14
V.K., Leurent, Volin '15

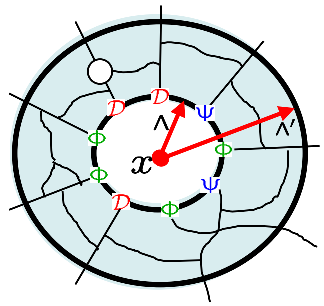
Dilatation operator in SYM perturbation theory

$$O_j(x) = \text{Tr}\left[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Phi\mathcal{F}\Psi\ldots\right](x)$$

- Expansion in orthonormal system of operators $\langle \mathcal{O}_A \mathcal{O}_B \rangle = \frac{\delta_{AB}}{(x^2)^{\Delta_A}}$

$$O_j = \sum_A C_{jA} \mathcal{O}_A$$

$$\langle O_j^\dagger(x) O_k(0) \rangle = \sum_{A,B} C_{jA}^* C_{kB} \frac{\delta_{AB}}{|x|^{2\Delta_A}} = \sum_A C_{Aj}^\dagger \frac{1}{|x|^{2\Delta_A}} C_{kA} = \langle j | \frac{1}{|x|^{2\hat{D}}} | k \rangle$$



- Dilatation operator $\hat{D}$ from point-splitting and renormalization

$$\mathcal{O}_j^{\Lambda'}(x) = \left[\left(\frac{\Lambda'}{\Lambda}\right)^{\hat{D}}\right]_{jk} \mathcal{O}_k^{\Lambda}(x)$$

- Conformal dimensions are eigenvalues of dilatation operator

$$\hat{D}_{jk} \mathcal{O}_k(x) = \Delta_j \mathcal{O}_j$$

- Can be computed from perturbation theory in coupling $g^2 = N g_{YM}^2$

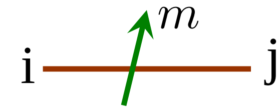
$$\hat{D} = \hat{D}^{(0)} + g^2 \hat{D}^{(2)} + g^4 \hat{D}^{(4)} + \ldots$$

$$\Delta = \Delta^{(0)} + g^2 \Delta^{(2)} + g^4 \Delta^{(4)} + \ldots$$

- Notice that

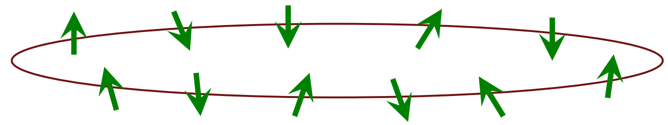
$$\frac{1}{|x|^{2\hat{D}}} \simeq \frac{1}{|x|^{2\hat{D}_0}} \left(1 - 2g^2 \log|x| \hat{D}^{(2)} + \ldots\right)$$

One loop dilatation: spin chain picture

- Example: SO(6) sector (scalar fields): $\Phi^{ij}_m \Longleftrightarrow$ 

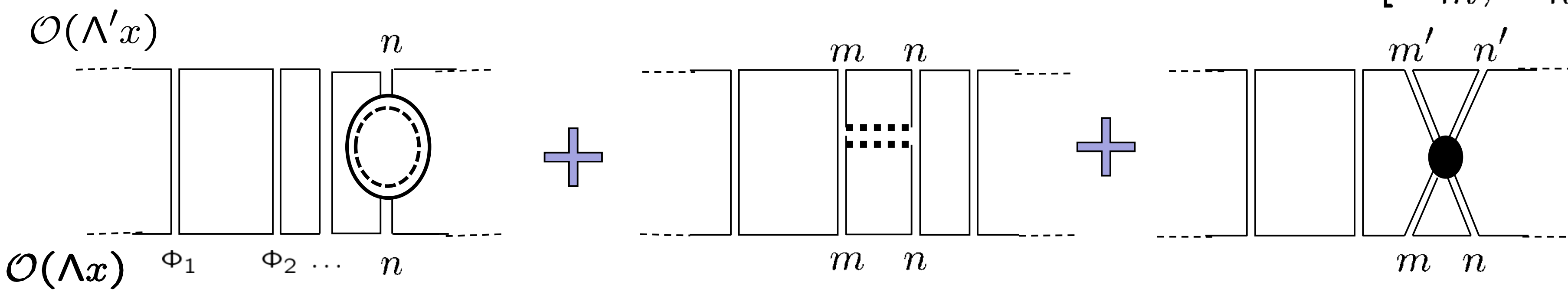
$m = 1, 2, \ldots, 6$

$$\mathcal{O}(x) = \mathrm{Tr}\left(\Phi_{n_1}\Phi_{n_2}\cdots\Phi_{n_L}\right)$$



- Tree level: $\Delta_0 = L$ - degeneracy (for scalars)
- 1-loop (examples of graphs):

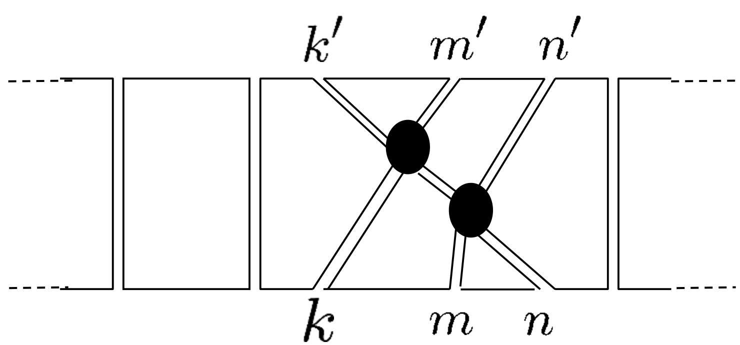
$\mathrm{Tr}\,[\Phi_m, \Phi_n]^2$



The first two diagrams do not change indices

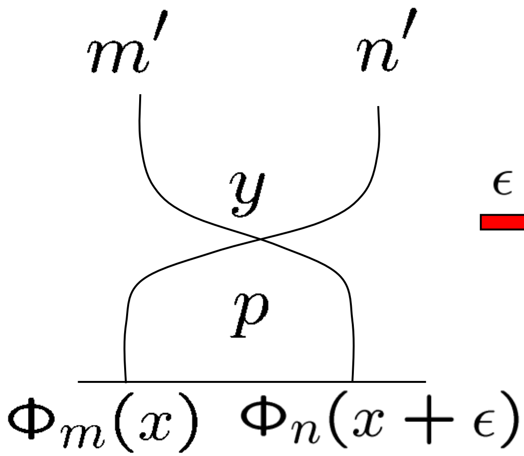
nontrivial action on R-indices.

- 2-loop:



Diagrams…..

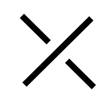
- Scalar vertex: $\frac{1}{2}\mathrm{Tr}\,[\Phi_a, \Phi_b]^2 = \mathrm{Tr}\,(\Phi_a \Phi_b)^2 - \mathrm{Tr}\,\Phi_a^2 \Phi_b^2$



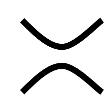
$$\xrightarrow{\epsilon \,\to\, \epsilon'} \quad \int \frac{d^4p \left(e^{i(p\cdot\epsilon)} - e^{i(p\cdot\epsilon')}\right)}{(2\pi)^4\, p^4} = \frac{1}{8\pi^2} \log \frac{\epsilon'}{\epsilon}$$

- SO(6) index structure: $\delta_{mn}\delta_{m'n'} - 2\delta_{mn'}\delta_{nm'} + 2\delta_{mm'}\delta_{nn'}$

Permutation (2nd term): $\mathcal{P} = \delta_{mn'}\delta_{nm'}$



Trace operator (1st term): $\mathcal{K} = \delta_{mn}\delta_{m'n'}$



Unity (3rd term): $\mathcal{I} = \delta_{mm'}\delta_{nn'}$

1-Loop Dilatation Operator

- Dilatation operator – integrable periodic SO(6) spin chain Hamiltonian:

Minahan, Zarembo '02

$$D = L + \frac{g^2}{16\pi^2} \sum_{l=1}^{L} \left(2\mathcal{I} + \mathcal{K}_{l,l+1} - 2\mathcal{P}_{l,l+1}\right) + O\left(\left(\lambda^2\right)\right), \qquad L+1 \to 1$$

- Defines the $6^L \times 6^L$ mixing matrix - a spin chain Hamiltonian.

- Minimum realized on BPS operators where we introduced complex scalars:

$$\mathcal{O}_{\text{BPS}} = \text{Tr } Z^L$$

$$X = \Phi_1 + i\Phi_2, \qquad Y = \Phi_3 + i\Phi_4, \qquad Z = \Phi_5 + i\Phi_6$$

- The coefficient of unity defined from SUSY: $\min D = L.$

- Integrability demands precisely these coefficients in D !

Full one loop dilatation operator

- Fundamental ("singleton") irrep of psu(2,2|4), with highest weight state (Z-field), is $\mathcal{V} = (1,0,0|1,0,0)$

$$\mathcal{V} = \{\mathcal{D}^n\Phi, \mathcal{D}^m\Psi, \mathcal{D}^k\mathcal{F}\} \qquad \{\mathcal{D}_{\dot\alpha\beta}, \Phi_{ab}, \Psi_{\alpha b}, \dot\Psi^b_{\dot\alpha}, \mathcal{F}_{\alpha\beta}, \dot{\mathcal{F}}_{\dot\alpha\dot\beta}\}$$

- Two neighbouring fields - direct product of singletons – expended into representations:

$$\mathcal{V}\otimes\mathcal{V} = \sum_{j=1}^{\infty} \mathcal{V}_j, \qquad \mathcal{V}_0 = (2,0,0|2,0,0), \quad \mathcal{V}_1 = (2,0,0|1,1,0), \quad \mathcal{V}_j = (j,j-2,j-2|0,0,0),\ j \geq 2$$

$$\text{projector} \quad \mathcal{P}^{(j)}_{l,l+1} \; : \quad \mathcal{V}^{(l)} \otimes \mathcal{V}^{(l+1)} \to \mathcal{V}_j$$

- 1-loop Dilatation operator for full psu(2,2|4) in terms of projector, transforming fields C,D to fields A,B on sites *l,l+1*

$$\mathcal{H}_{l,l+1} \equiv (D-D_0)_{l,l+1} = 2\lambda \sum_{j=1}^{\infty} h(j) \left(\mathcal{P}_j\right)^{AB}_{CD} : \mathrm{tr}\,[\chi_A, \check\chi_B]\,[\chi_C, \check\chi_D] :, \qquad (\check\chi \equiv \frac{\partial}{\partial\chi})$$

matrix derivative

- Examples:

harmonic numbers

$$\mathcal{H}_{l,l+1}(ZX) = \mathrm{tr}([Z,X][\check Z,\check X]), \quad \mathcal{H}_{l,l+1}(\psi\psi) = -2\mathrm{tr}(\psi\psi\check\psi\check\psi), \qquad h(j) = \sum_{k=1}^{j}\frac{1}{k}$$

$$\mathcal{H}_{l,l+1}(\mathcal{F}\psi) = \frac{5}{2}\mathrm{tr}([\mathcal{F},\check{\mathcal{F}}]\{\psi\check\psi\}) - \frac{1}{2}\mathrm{tr}([\psi\check{\mathcal{F}}][\mathcal{F},\check\psi]), \ \text{etc}$$

Perturbative Integrability (su(2)-sector: only Z,X)

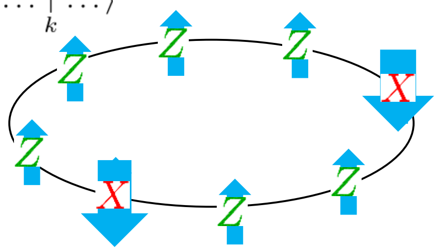
- 1-loop dilatation operator = Hamiltonian of Heisenberg quantum spin chain. Integrable by Bethe ansatz! SU(2) Hamiltonian (projection of SO(6)):

$$\hat{D} = L + g^2\, 2 \sum_{l=1}^{L} \left(1 - \mathcal{P}_{l,l+1}\right)$$

permutation

$$\mathcal{P}_{j,k} | \ldots \underset{j}{\uparrow} \ldots \underset{k}{\downarrow} \ldots \rangle = | \ldots \underset{j}{\downarrow} \ldots \underset{k}{\uparrow} \ldots \rangle$$

$\mathrm{Tr}\, Z^L(x)$ - vacuum



also integrable!

Beisert, Kristijansen,Staudacher

Solution for spectrum (diagonalization of *D*):

$$q^L \left(\frac{u_k + i/2}{u_k - i/2} \right)^L = \prod_{(k\neq)j=1}^{J} \frac{u_k - u_j + i}{u_k - u_j - i}$$

Bethe equations

Bethe'31

Trace cyclicity condition:

$$q^L \prod_{k=1}^{J} \frac{u_k + i/2}{u_k - i/2} = 1$$

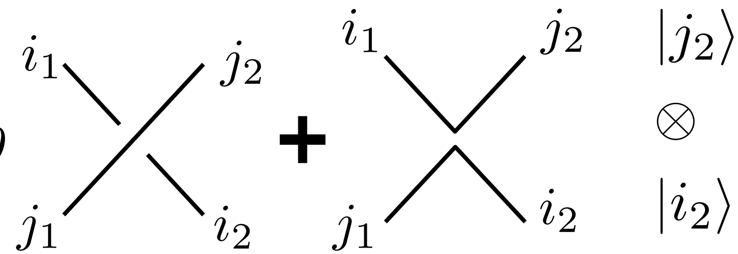
Anomalous dimension = Energy for spin chain H:

$$\Delta - L = g^2 \sum_{k=1}^{J} \frac{1}{u_k^2 + 1/4}$$

Basics of integrability for SU(N) spin chain

- R-matrix $\hat{R} = i\,\theta\,\mathbf{1}\times\mathbf{1} + \mathcal{P} =$

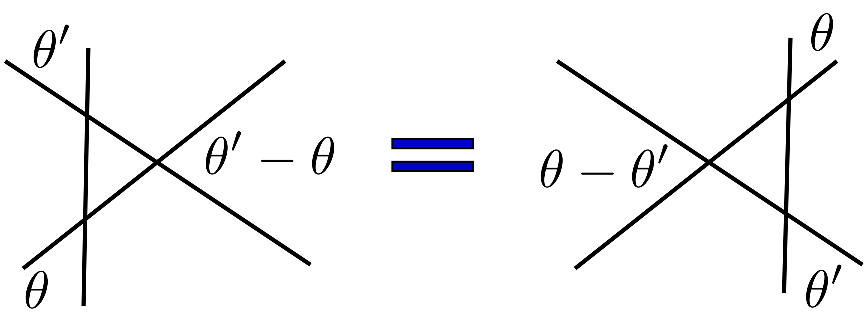
$$= i\,\theta\,\delta^{i_1}_{i_2}\,\delta^{j_1}_{j_2} + \delta^{i_1}_{j_2}\,\delta^{j_1}_{i_2} = \; i\,\theta$$



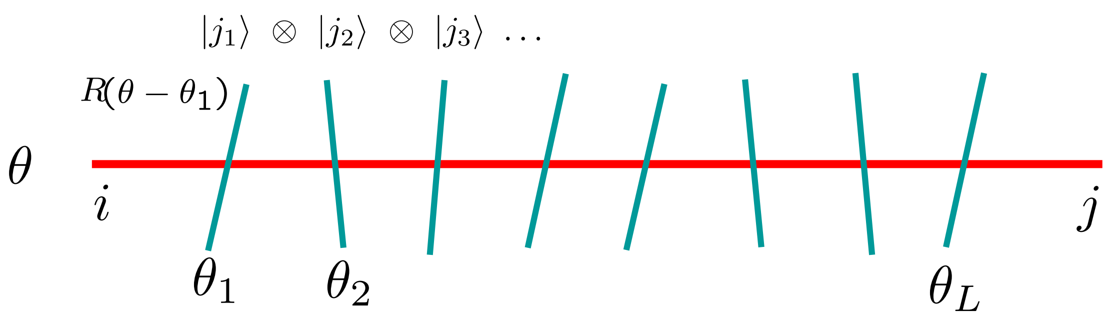
$i, j = 1, 2, \ldots, N$

$\boxed{i \;\text{———}\; j \; = \delta_{ij}}$

- Yang-Baxter equation $R(\theta)R(\theta')R(\theta'-\theta) = R(\theta-\theta')R(\theta')R(\theta)$



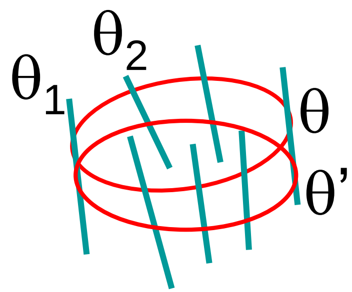
- Monodromy matrix $\mathcal{T}(\theta|\theta_1,\ldots,\theta_L) = R(\theta-\theta_1)\ldots R(\theta-\theta_L)$



Commutativity of T-matrices using Yang-Baxter

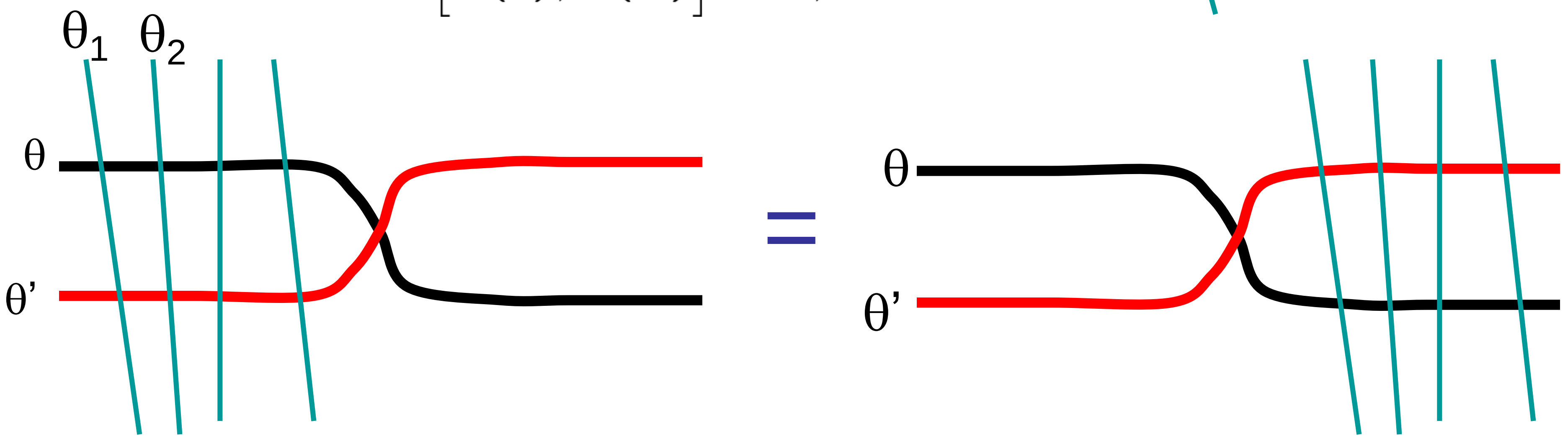
T-matrix is a polynomial of degree L

$$\hat{T}(\theta) = \mathrm{tr}_{\mathrm{aux}}\ \mathcal{T}(\theta)$$



Integrability!

$$\left[\hat{T}(\theta), \hat{T}(\theta')\right] = 0,$$



$$\mathcal{T}(\theta)\mathcal{T}(\theta')R(\theta' - \theta) = R(\theta - \theta')\mathcal{T}(\theta')\mathcal{T}(\theta)$$

- We prove commutativity taking traces on both aux spaces and inspecting two terms in R-matrices on both sides. One term gives trivial identity, another term - commutativity

Hamiltonian from T-matrix energy from Q-function

- Twisted Hamiltonian

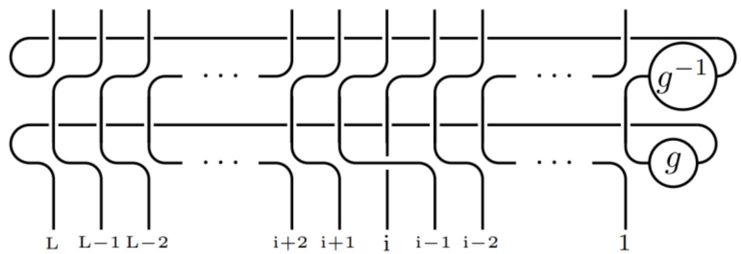
$$H = \sum_{i=1}^{L-1} \mathcal{P}_{i,i+1} + \mathcal{P}_{L,1}\; g_L g_1^{-1}$$

$$g = U^{-1}\text{diag}\{x_1, \cdots, x_N\}U \quad \in \quad SU(N)$$

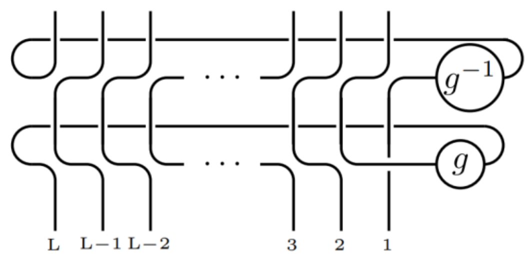
- Proof that the Hamiltonian is one of the conservation laws (θ=u, θ_1= θ_2=…= 0)

$$\left.\frac{\partial_u T(u)}{T(u)}\right|_{u=0} = \Big(T(0)\Big)^{-1} \cdot \partial_u T(u)|_{u=0}$$

$$= \sum_{i=1}^{L-1}$$



$$+$$



$$= \sum_{i=1}^{L-1} \mathcal{P}_{i,i+1} + \mathcal{P}_{1,L} \cdot g_L \cdot g_1^{-1}\,.$$

Diagonalization by Algebraic Bethe Ansatz: SU(2)

Leningrad school (LOMI) '80s

- Right monodromy matrix with operator entrees:

$$\hat{\mathcal{T}}(\theta) = \prod_{k=1}^{L} R_{0k}(\theta - \theta_k) = \begin{pmatrix} \hat{A}(\theta) & \hat{B}(\theta) \\ \hat{C}(\theta) & \hat{D}(\theta) \end{pmatrix}$$

- Transfer matrix $\quad T = \mathrm{tr}\mathcal{T} = \hat{A} + \hat{D}$

- Yang-Baxter eq. reads (in explicit indices)

$$\mathcal{T}^{ab}(\theta)\,\mathcal{T}^{a'b'}(\theta')\,{}_{bb'}R_{cc'}(\theta'-\theta) = {}_{aa'}R_{bb'}(\theta'-\theta)\,\mathcal{T}^{bc}(\theta')\,\mathcal{T}^{b'c'}(\theta)$$

or in components,

$$\begin{aligned}
[B(\theta), B(\theta')] &= 0 \\
A(\theta)\,B(\theta') &= \frac{\theta'-\theta-i}{\theta'-\theta}\,B(\theta')\,A(\theta) + \frac{i}{\theta'-\theta}\,B(\theta)\,A(\theta') \\
D(\theta)\,B(\theta') &= \frac{\theta'-\theta+i}{\theta'-\theta}\,B(\theta')\,D(\theta) - \frac{i}{\theta'-\theta}\,B(\theta)\,D(\theta')\,.
\end{aligned}$$

*

Construction of eigenstate

- A state $|\Psi(u_1,\ldots,u_J\,|\,\theta_1,\ldots,\theta_L)\rangle = \prod_{j=1}^{J} \hat{B}(u_j)|\,\Omega\rangle$

  where Ω=|↑ ↑ ↑… ↑ › is the highest weight state (all isospins up).

$$R_{0k}(\theta)|\uparrow\rangle = \begin{pmatrix} (\theta - i)|\uparrow\rangle & * \\ 0 & \theta|\uparrow\rangle \end{pmatrix}$$

- This means that Ω diagonalizes $\hat{T}(\theta) = \hat{A}(\theta) + \hat{D}(\theta)$

$$A(\theta)|\,\Omega\rangle = \prod_{\alpha=1}^{L}(\theta - \theta_\alpha - i)\,|\,\Omega\rangle$$

$$D(\theta)|\,\Omega\rangle = \prod_{\alpha=1}^{L}(\theta - \theta_\alpha)\,|\,\Omega\rangle$$

Construction of a state diagonalizing T-matrix

- Action on general state $|\Psi\rangle = |\{u_j\}\rangle$ using Yang-Baxter relations

$$\hat{T}(\theta)\,|\Psi\rangle = (\hat{A}(\theta)+\hat{D}(\theta)) \times \prod_{j=1}^{J} \hat{B}(u_j)|\Omega(\theta_1,\dots,\theta_L)\rangle$$
$$= \left(\prod_{\alpha=1}^{L}(\theta-\theta_\alpha-i)\prod_{j=1}^{J}\frac{u_j-\theta-i}{u_j-\theta} + \prod_{\alpha=1}^{L}(\theta-\theta_\alpha)\prod_{j=1}^{J}\frac{u_j-\theta+i}{u_j-\theta}\right)|\Psi\rangle + \dots$$

- The (unwanted) $B(\theta)A(u)$, $B(\theta)D(u)$ terms should vanish.
  It can be assured by analyticity property: $T(\theta)$ is a polynomial
  by construction. Cancelling the poles at $\theta=u_i$ we obtain, after shifting $u_a \to u_a + \frac{i}{2}$
  the Bethe ansatz equation for u_i's

$$1 = \prod_{\beta}^{L}\frac{u_j-\theta_\beta-i/2}{u_j-\theta_\beta+i/2}\prod_{i\neq j}^{J}\frac{u_j-u_i+i}{u_j-u_i-i}$$

- For our spin chain Hamiltonian, all $\theta_j = 0$

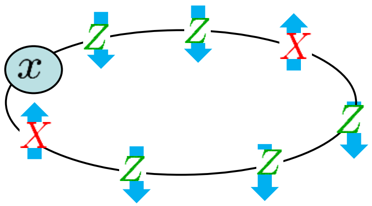
- Energy of a state for a given solution of BAE $\{u_i\}$

$$E = \sum_{k=1}^{J}\frac{1}{u_k^2+1/4}$$

Q-system for Heisenberg spin chain

- Hamiltonian with twist $$H=\sum_{i=1}^{L-1}\mathcal{P}_{i,i+1}+\mathcal{P}_{L,1}\ g_L g_1^{-1}$$

twist $$g=\begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}\in SU(2)$$



- Baxter TQ relation (from AlgBA):

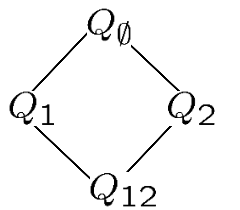
$$T(u)Q(u)=Q(u+i)\,(u-i/2)^L+Q(u-i)\,(u+i/2)^L$$

- SU(2) Q-functions for J magnons:

$$Q_1(u)=x^{iu}\prod_{j=1}^{J}(u-u_j),\qquad Q_2(u)=x^{-iu}\prod_{j=1}^{L-J}(u-\tilde{u}_j)$$

- Wronskian condition from TQ (QQ-relation)

$$Q_{12}(u)\equiv\begin{vmatrix} Q_1(u+i/2) & Q_1(u-i/2) \\ Q_2(u+i/2) & Q_2(u-i/2)\end{vmatrix}=\left(x-\frac{1}{x}\right)\ u^L$$



is equivalent to BAE which give L!/((L-J)!J!) solutions of BAE:

Hasse diagram

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^L=-\frac{Q_1(u_k+i)}{Q_1(u_k-i)},\qquad k=1,2,\cdots,J$$

- (non)-zero momentum trace (non)-cyclicity) condition xT(θ=-i/2)=1: $$x\frac{Q(i/2)}{Q(-i/2)}=1$$

- Energy: $$\Delta-L=g^2\,i\partial_u\log\frac{T(u)}{u^L}\Big|_{u=0}=g^2\,i\,\partial_u\log\left(\frac{Q(u+i/2)}{Q(u-i/2)}\right)\Big|_{u=i/2}$$ $Q=Q_1$ or Q_2

- When untwisting x → 1 one has to increase powers of Q's.

Q-system for SU(N) (without proof)

- Hamiltonian with twist $\qquad g = U^{-1}\text{diag}\{x_1, \cdots, x_N\}U \quad \in \quad SL(N)$

$$H = \sum_{i=1}^{L-1} \mathcal{P}_{i,i+1} + \mathcal{P}_{L,1}\ g_L g_1^{-1}$$

permutation

$$\mathcal{P}_{2,4}|\uparrow\swarrow\nwarrow\uparrow\nearrow\ \ldots\rangle = |\uparrow\uparrow\nwarrow\swarrow\nearrow\ \ldots\rangle$$

- SU(N) Q-functions:

$$Q_k(u) = x_k^{iu}\prod_{j=1}^{J_k}(u-u_j^{(k)}), \quad k = 1, 2, \ldots, N\,, \qquad \sum_{k=1}^{N} J_k = L$$

Cartan charges

$$\{J_1, J_2, \cdots, J_N\}$$

- Wronskian condition (amounts to nested BAE's)

$$Q_{12\cdots N}(u) \equiv \text{Det}_{\ 1\le k,j\le N} Q_k^{[N-2j+1]} = \Delta(x_1, \cdots, x_N)u^L$$

Vandermonde determinant $\qquad \Delta(x_1, \cdots, x_N) = \prod_{k>j}(x_k - x_j)$

notations:

$$f^{[n]} \equiv f(u + \frac{in}{2})$$

$$f^{\pm} \equiv f(u \pm \frac{i}{2})$$

- Energy (true for any of N Q-functions):

$$\Delta - L \stackrel{\cdot}{=} g^2\partial_u \log T(u)\,\big|_{u=i/2} = ig^2\,\partial_u \log\frac{Q_k(u-i/2)}{Q_k(u+i/2)}\,\big|_{u=0} = g^2\sum_{j=1}^{J_k}\frac{1}{u_j^{(k)2}+1/4}$$

Q-system as a Grassmanian

Krichever,Lipan, Wiegmann,Zabrodin
Gromov, Vieira
V.K., Leurent, Volin.

- One-form on N single indexed Q-functions:

$$Q_{(1)} \equiv \sum_{j=1}^{N} Q_j(u)\xi^j, \qquad \{\xi^i,\xi^j\}=0$$

- l-form encodes all Q-functions with l indices:

$$Q_{(l)} \equiv Q_{(1)}^{[-l+1]} \wedge Q_{(1)}^{[-l+3]} \wedge \ldots \wedge Q_{(1)}^{[l-1]}$$

- Multi-index Q-function: coefficient of $\xi_{i_1} \wedge \xi_{i_2} \wedge \ldots \wedge \xi_{i_l}$

$$Q_I \equiv Q_{j_1,\ldots,j_k} = \det_{1\le m,n\le k} Q_{j_m}^{[-1-k+2n]}, \qquad I=\{j_1,\ldots,j_k\}\subset\{1,2,\ldots,N\}$$

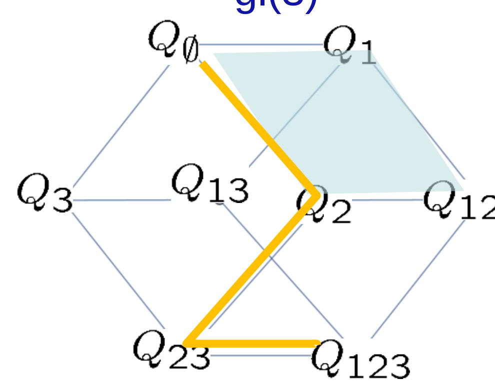
- Example N=2: $Q_{(2)} = 2Q_{12}\,\xi_1\wedge\xi_2, \qquad Q_{12} = Q_1(u+\frac{i}{2})Q_2(u-\frac{i}{2}) - Q_1(u-\frac{i}{2})Q_2(u+\frac{i}{2})$

- Notations in terms of subsets of indices:

$$\mathrm{Q}_{j_1,\ldots,j_k} \equiv \mathrm{Q}_I\,, \qquad I=\{j_1,\ldots,j_k\}\subset\{1,2,\ldots,N\}$$

- Plücker's QQ-relations: $Q_I Q_{I,i,j} = Q_{I,i}^{+}Q_{I,j}^{-} - Q_{I,i}^{-}Q_{I,j}^{+}$
- Hasse diagram (N-dim. hypercub): QQ-rel. $\leftrightarrow$ faces
- BAE: write QQ relations along "nesting path" and impose polynomiality
- Wronskian form of BAE: $Q_{\text{Full set}} \equiv Q_{123\ldots N} = \Delta(x)u^L$

gl(3)



Conclusions

- Dilatation operator of N=4 SYM identified as a Hamiltonian of a1d chain.
- Twisted b.c. possible
- The coefficients in the Hamiltonian are precisely the ones imposed by integrability, even in higher loops!
- Techniques of Bethe ansatz come into play.
- Integrability of string sigma model is important (absent in these lectures)
- Solution in all loops given by Quantum Spectral Curve (next lecture)

References:

- J.Minahan, https://arxiv.org/pdf/1012.3983.pdf
- S.Leurent, https://arxiv.org/pdf/1206.4061.pdf (thesis, ch.I,II)
- L. Faddeev, https://arxiv.org/pdf/hep-th/9605187.pdf
- Korepin, V.E.; Bogoliubov, N.M.; Izergin, A.G. (1997). *Quantum Inverse Scattering Method and Correlation Functions*. Cambridge University Press.

Lecture III

All-loop integrability of AdS/CFT spectrum:

Quantum Spectral Curve (QSC)

QSC of N=4 SYM

- The AdS/CFT QSC allows to compute, in the planar limit, the exact anomalous dimension $\Delta(g)$ of any single trace operator with quantum numbers $(S_1,S_2|J_1,J_2,J_3)$.
- QSC is a system of non-linear, grassmanian algebraic equations - Q-system, on 2^8=256 Q-functions (generalisation of Baxter's Q-functions), supplied by analyticity conditions w.r.t. spectral parameter u.
- The algebraic relations of Q-system are conveniently encoded into the Hasse diagram. The algebraic structure of Q-system is closely related to the one of the gl(4|4) supersymmetric Heisenberg spin chain.
- The analyticity conditions are very different from the spin chain analyticity (but resembling it in the perturbation theory). This analyticity is rather of the 2d QFT type and it was established with the help of Metsaev-Tseytlin string sigma-model.
- QSC represents a finite system of non-linear Riemann-Hilbert equations finding its origins in the Thermodynamical Bethe Ansatz. No formal proof is available!
- QSC is the most efficient way to compute dimensions of particular operators: in the week coupling (well established), numerically (very efficient!), in strong coupling (still not very developed). Still a complex tool, needs deep knowledge…

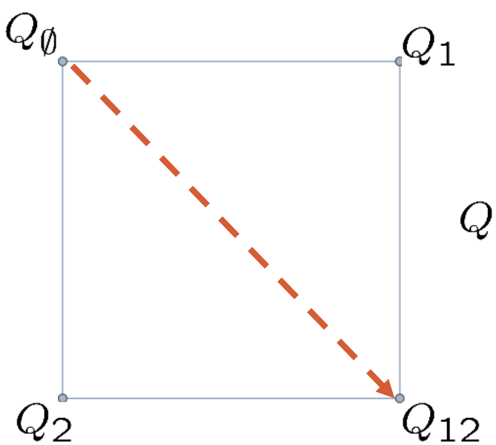
Quantum Spectral Curve of $\mathcal{N}$=4 SYM: algebraic structure

Gromov, V.K., Leurent, Volin '13,'14
V.K., Leurent, Volin '15

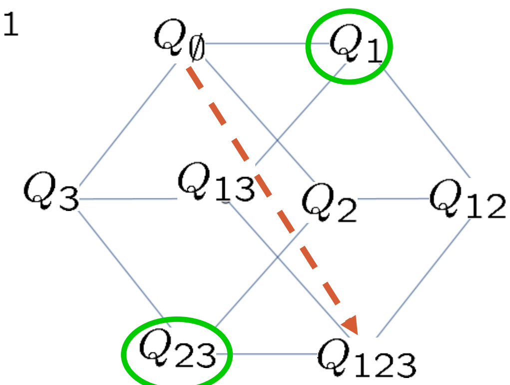
- QSC eqs. close on finite set of Baxter functions of spectral parameter $Q_I(u)$
- gl(n): each Q placed at a vertex of Hasse diagram - n-hypercube

for $\mathcal{N}$=4 SYM

gl(2)

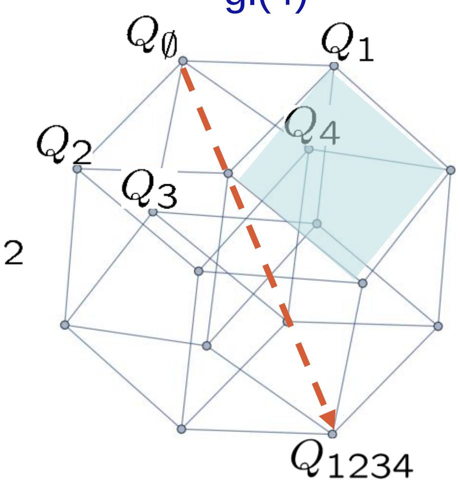


gl(3)

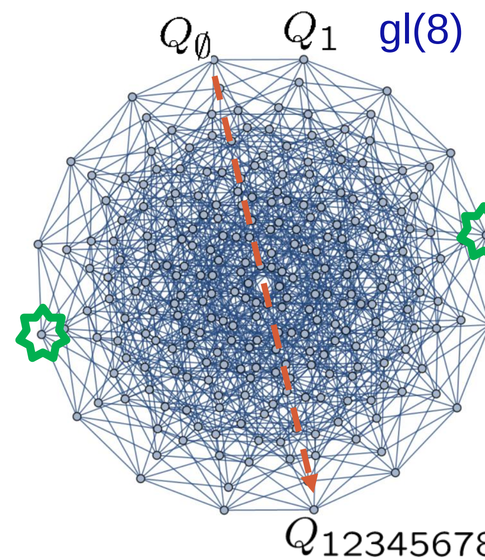


SUSY: gl(1|2)

gl(4)

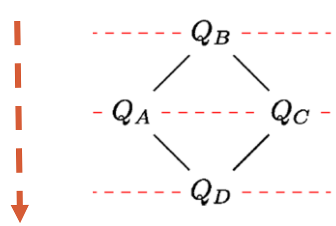


gl(8)



SUSY: gl(8) → gl(4|4)

- Plücker QQ-relations on each face ("determinant flow"):



$$\Leftrightarrow \quad Q_B(u)Q_D(u) = \begin{vmatrix} Q_A(u+\frac{i}{2}) & Q_C(u+\frac{i}{2}) \\ Q_A(u-\frac{i}{2}) & Q_C(u-\frac{i}{2}) \end{vmatrix}$$

Grassmanian structure!

Krichever, Lupan, Wiegmann, Zabrodin, '94
Tsuboi '13

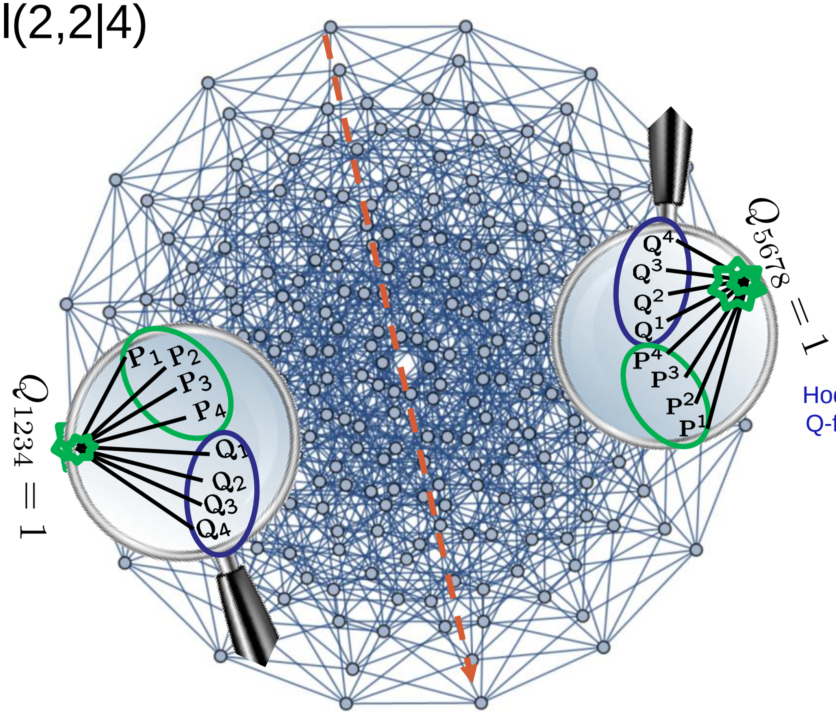
- gl(N) Heisenberg spin chain: $Q_0 = 1, \quad Q_j = \text{Polynomial}(u), \quad Q_{1,2,\dots,N} = \det_{k,j} Q_k(u+ij) \sim u^{\text{Length}}$
- gl(M|K) Heisenberg spin chain: $Q_{12\dots M} = 1, \quad Q_{\text{neighbors}\,12\dots M} = \text{Polyn}(u) \quad Q_{M+1,M+2,\dots K} \sim u^{\text{Length}}$

Gromov, V.K., Leurent, Volin '13,'14
V.K., Leurent, Volin '15

Quantum Spectral Curve of AdS_5/CFT_4: analytic structure

- Choose special 8 + 8 Q-functions with nice analyticity on physical sheet

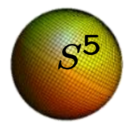
gl(2,2|4)



Hodge dual Q-functions

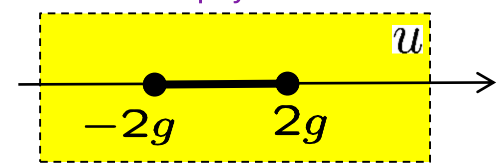
Large u asymptotics fixed by PSU(2,2|4)

Cartan charges $\{\Delta, S_1, S_2 | J_1, J_2, J_3\}$
SO(2,4) SO(6) ~ SU(4)



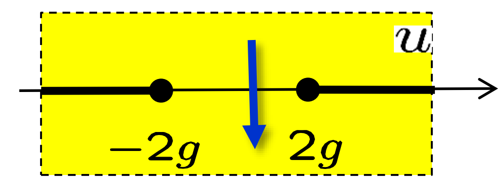
$$\mathbf{P}_b,\ \mathbf{P}^b \sim q_b^{\pm iu}\, u^{(\pm J_1 \pm J_2 \pm J_3)/2}$$

short cut on physical sheet



$$\mathbf{Q}_j,\ \mathbf{Q}^j \sim u^{(\pm\Delta \pm S_1 \pm S_2)/2}$$

long cut on physical sheet



- Various Q-functions are related by complex conjugation ("gluing" relations)

$$\mathbf{Q}_1 \propto \bar{\mathbf{Q}}^2\,, \quad \mathbf{Q}_2 \propto \bar{\mathbf{Q}}^1\,, \quad \mathbf{Q}_3 \propto \bar{\mathbf{Q}}^4\,, \quad \mathbf{Q}_4 \propto \bar{\mathbf{Q}}^3$$

- These Riemann-Hilbert conditions fix all physical solutions for Q-system and thus conformal dimensions $\Delta(g)$ with given PSU(2,2|4) charges

Algebraic symmetries of AdS/CFT Q-system

- **Hodge duality** is a simple relabeling:

$$Q^{\{A\}} \equiv Q_{F\backslash A}\,, \qquad A \subset F \equiv \{12345678\}$$

Example for (4|4): $Q^{13467} \equiv Q_{258}$

- Hodge duals satisfy the same QQ-relations if we impose:

$$Q_{1234} \equiv Q^{5678} = Q_{5678} = 1$$

(related to super-unimodularity of PSU(2,2|4))

- **H-symmetry**: sl(4) × sl(4) rotation with i-periodic H-matrices preserving QQ-relations.

$$Q_{kj\ldots} \to (H_k^{k'} H_j^{j'} \ldots)^{\text{shift}} Q_{k'j'\ldots}\,, \qquad H_k^{k'}(u+i) = H_k^{k'}(u)$$

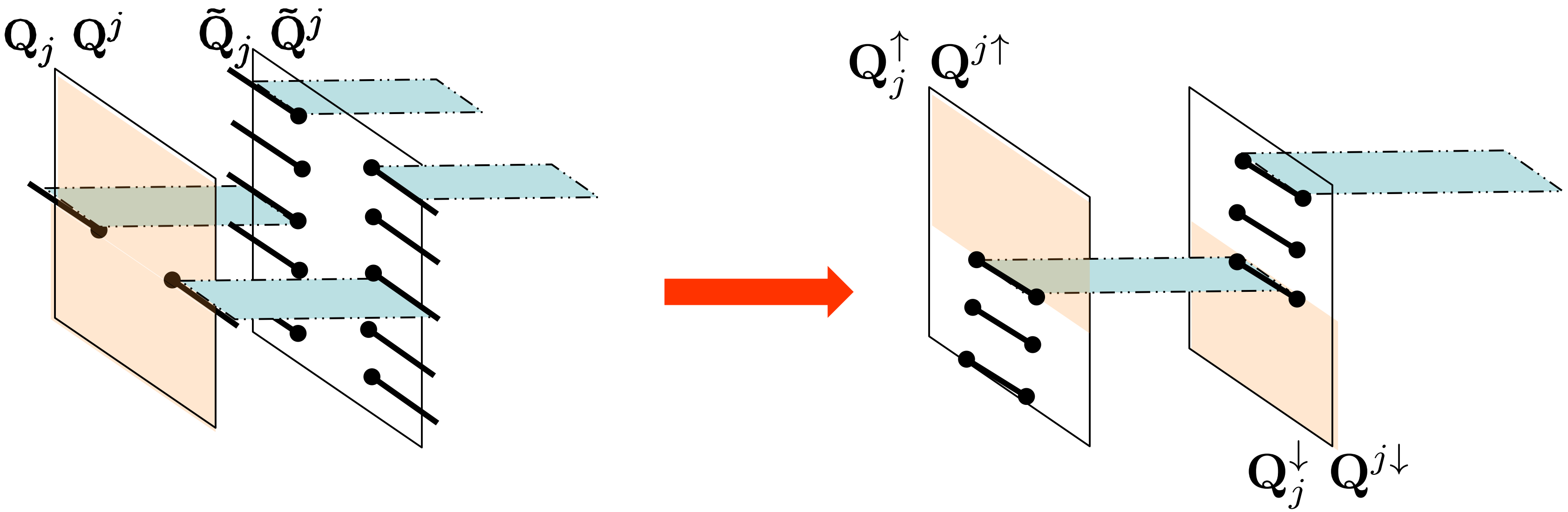
where the 8 × 8 H-matrix is block-diagonal, with 4 × 4 blocks:

$$H = \begin{pmatrix} H_b & 0 \\ 0 & H_f \end{pmatrix}$$

"shift" corresponds to all $H(u) \to H(u+i/2)$ for even number of indices (no shift for odd)

H(ω)-transformation from upper- to lower-analytic Q's

- Structure of cuts of Q-functions:



- We can "flip" all short cuts to long ones going through the short cuts from above or from below. It gives the upper or lower-analytic Q's.
- Q-system allows to choose all Q-functions upper-analytic or all lower-analytic. Both representations are physically equivalent → related by H-rotations with periodic coefficients rising and lowering indices $\hat{\omega}_{ab}(u+i) = \hat{\omega}_{ab}(u)$

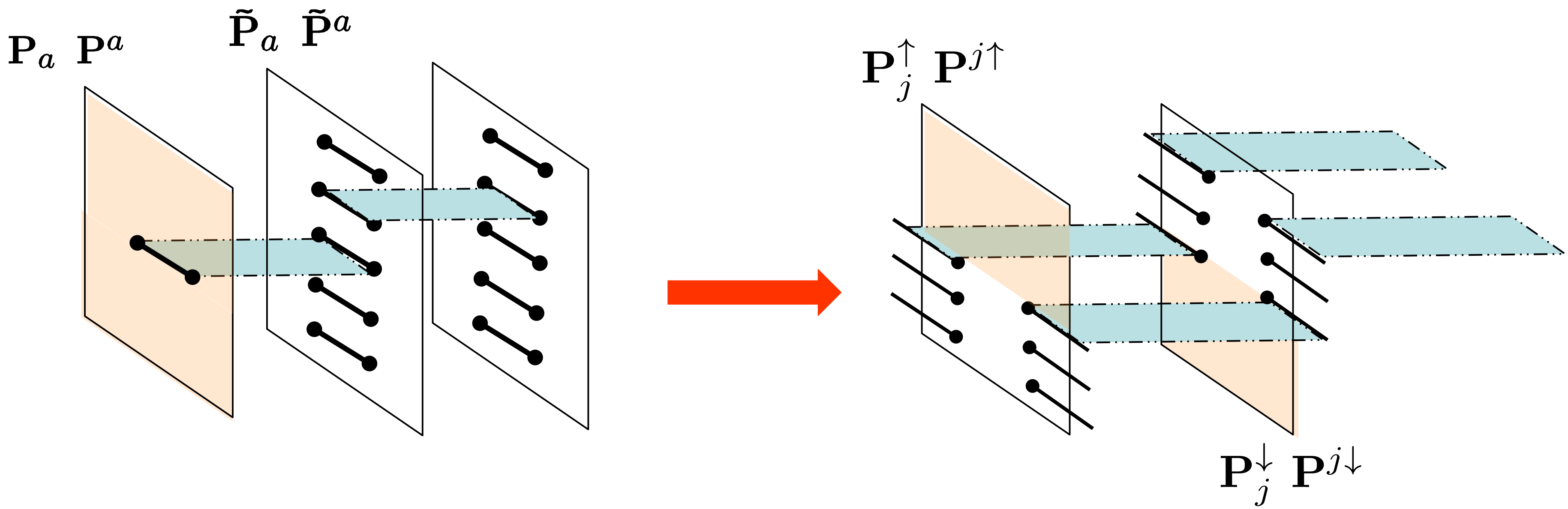
$$\mathbf{Q}_j^{\uparrow} = \hat{\omega}_{jk}\mathbf{Q}^{k\downarrow} \qquad\qquad \mathbf{Q}^{j\downarrow} = \hat{\omega}^{jk}\mathbf{Q}_k^{\uparrow}$$

$$\omega^{ij} = (\omega)^{-1}_{kl} = -\frac{1}{2}\epsilon^{ijkl}\omega_{kl}, \qquad \mathrm{Pf}(\omega) = 1$$

- True only for 4×4 antisym. matrices: Exceptional role of PSU(2,2|4) !

H(μ)-transformation from upper- to lower-analytic Q's

- Structure of cuts of P-functions: the same picture, but with the exchange of roles of long and short cuts



- Upper-analytic or all lower-analytic functions with long cuts related by H-rotation with periodic coefficients rising and lowering indices: $\check{\mu}_{ab}(u+i) = \check{\mu}_{ab}(u)$

$$\mathbf{P}_a^{\downarrow} = \check{\mu}_{ab}\mathbf{P}^{b\uparrow} \qquad \mathbf{P}^{a\uparrow} = \check{\mu}^{ab}\mathbf{P}_b^{\uparrow}$$

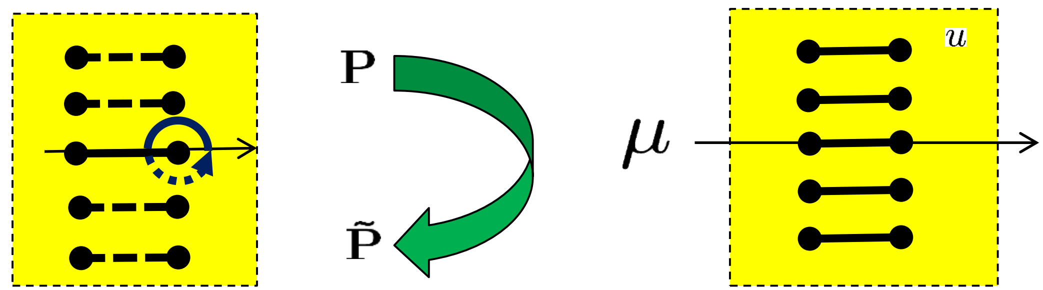
$$\check{\mu}^{ab}\check{\mu}_{bc} = \delta^a_c$$

Pμ-system

- H(μ) transformation defines monodromy through short cuts:

$$\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b \qquad\qquad \tilde{\mu}_{ab}(u) = \mu_{ab}(u+i)$$

$\tilde{\mathbf{P}}$ is the analytic continuation of $\mathbf{P}$ through the cut:



- Pμ -system containts the equation for μ (follows from QQ-relations):

$$\mu_{ab}(u+i) - \mu_{ab}(u) = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a \implies \mu_{ab}^{++} - \mu_{ab} = \mathbf{P}_a\mathbf{P}^c\mu_{bc} - \mathbf{P}_b\mathbf{P}^c\mu_{ac}$$

- Excluding from Pμ system $\tilde{\mathbf{P}}$ we get a linear eq. on μ similar to a QQ relation. So we interpret μ as a linear combination of 6 solutions of the last equation with short cuts, with i-periodic coefficients packed into antisymmetric ω-matrix

$$\mu_{ab} = -\frac{1}{2}Q^-_{ab,4+i,4+j}\,\omega^{ij}\,, \qquad a,b,i,j \in \{1,2,3,4\} \qquad\qquad \omega^{ij}(u) = \omega^{ij}(u+i)$$

- Similar Riemann-Hilbert eqs with long cuts can be written on $\mathbf{Q}_j$ and ω_{ij}

$$\tilde{\mathbf{Q}}_j = \omega_{jk}\mathbf{Q}^k \qquad\qquad \omega_{jk}(u+i) - \omega_{jk}(u) = \mathbf{Q}_k\tilde{\mathbf{Q}}_j - \mathbf{Q}_j\tilde{\mathbf{Q}}_k$$

Gromov, V.K., Leurent, Volin 2013

Pμ-system: example of SL(2) sector $\mathrm{Tr}(\nabla^S Z^L)$

- Numerous applications of quantum spectral curve to various sectors!
  Notorious approximations: weak coupling λ → 0 and BFKL (Regge limit) S → -1

- SL(2) – example of "Left-Right symmetric" sector su(2|2)$_L$ ↔ su(2|2)$_R$

$$\mu^{-1} = \chi\mu\chi$$
$$\mu_{23} = \mu_{14}$$

$$\chi = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Reduction: $\mathbf{P}^a = -\chi^{ab}\mathbf{P}_b \qquad \mathbf{Q}^a = -\chi^{ab}\mathbf{Q}_b$

- From formulation of QSC: symptotics at $u \longrightarrow \infty$

$$\begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{pmatrix} \simeq \begin{pmatrix} A_1 u^{-\frac{L}{2}} \\ A_2 u^{-\frac{L+2}{2}} \\ A_3 u^{\frac{L}{2}} \\ A_4 u^{\frac{L-2}{2}} \end{pmatrix} \qquad \begin{pmatrix} \mu_{12} \\ \mu_{13} \\ \mu_{14} \\ \mu_{24} \\ \mu_{34} \end{pmatrix} \sim \begin{pmatrix} u^{\Delta-L} \\ u^{\Delta+1} \\ u^{\Delta} \\ u^{\Delta-1} \\ u^{\Delta+L} \end{pmatrix}$$

Example: SL(2) sector at one loop from P-μ

- Plugging these asymptotics into Pμ eq. we get for coefficients of asymptotics

$$A_1 = -\frac{[(L+S-2)^2-\Delta^2][(L-S)^2-\Delta^2)]}{16L(L-1)}, \qquad A_2 = -\frac{[(L-S+2)^2-\Delta^2][(L+S)^2-\Delta^2)]}{16L(L+1)}$$

$$A_3 = A_4 = 1$$

- In weak coupling, since we know that $\Delta = L + S + g^2\gamma + \mathcal{O}(g^4)$
we can put $\mathbf{P}_2 \simeq 0 + \mathcal{O}(g^2)$ and the system of 5 equations on μ_{ab}
reduces to one 2-nd order difference equation on $\mu_{12}^+ \equiv Q + \mathcal{O}(g^2)$

$$T\,Q + \frac{1}{(\mathbf{P}_1^-)^2}Q^{--} + \frac{1}{(\mathbf{P}_1^+)^2}Q^{++} = 0, \text{ where } \quad T = \frac{\mathbf{P}_4^+}{\mathbf{P}_1^+} - \frac{\mathbf{P}_4^-}{\mathbf{P}_1^-} - \frac{1}{(\mathbf{P}_1^-)^2} - \frac{1}{(\mathbf{P}_1^+)^2}$$

- We can argue that $\mathbf{P}_1 = A_1 u^{-L/2} + \mathcal{O}(g^2)$ (coincides with asymptotics) since

$$|u| \ll g \ll 1$$

$$P_1 = \frac{P_1 + \tilde{P}_1}{2} + \sqrt{u^2-4g^2}\left(\frac{P_1 - \tilde{P}_1^{++}}{2\sqrt{u^2-4g^2}}\right) \simeq \mathrm{reg}_1(u) + \left[u - \frac{2g^2}{u} + O(g^4/u^3)\right]\mathrm{reg}_2(u)$$

- No singularities in $\mathbf{P}_4/\mathbf{P}_1$ at finite u and no pole at u=0 in the leading order in g

$$\frac{\mathbf{P}_4}{\mathbf{P}_1} = c_{L-1}u^{L-1} + c_{L-2}u^{L-1} + \cdots + c_0 + \mathcal{O}(g^2)$$

So T is a polynomial! We get standard Baxter eq. for SL(2) Heisenberg spin chain!

$$T(u)Q(u) = \left(u + \frac{i}{2}\right)^L Q(u+i) + \left(u - \frac{i}{2}\right)^L Q(u-i)$$

where

$$\mu_{12}^+ \simeq Q(u) = \prod_{k=1}^{S}(u - u_k)$$

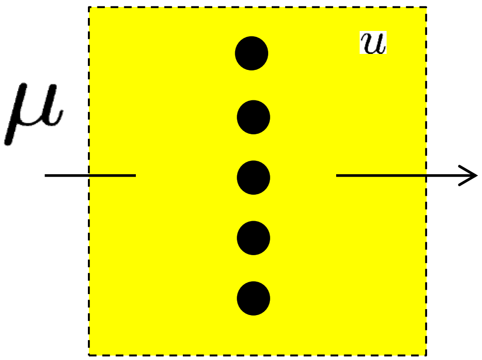
- Cyclicity from absence of the cut at $g = 0$

$$\mu_{12}^{++} = \tilde{\mu}_{12} = \mu_{12} \quad \Rightarrow \quad Q^+ = Q^-$$

One loop anomalous dimensions for SL(2)

- To find anomalous dimensions demands the solution of P-µ system to the next order in $\mu_{12}^{+} \equiv Q + \mathcal{O}(g^2)$ as seen from asymptotics

$$\mu_{12} \sim u^{\Delta - L} \sim u^S(1 + g^2\gamma \log u)$$



- We split μ_{12} into regular and singular parts at each cut

$$\mu_{12} = \frac{\mu_{12} + \mu_{12}^{++}}{2} + \sqrt{(u - in)^2 - 4g^2}\left(\frac{\mu_{12} - \mu_{12}^{++}}{2\sqrt{(u - in)^2 - 4g^2}}\right)$$

- At $|u| \ll g \ll 1$ we get poles: $\sqrt{(u - in)^2 - 4g^2} \simeq u - in - 2\frac{g^2}{u - in} + O(g^4)$
- For singular part at poles:

$$\mu_{12}^{sing}(u + i/2) = f(u) \sum_{n \in \mathbb{Z}} \frac{\mathrm{sgn}(n - 1/2)}{iu - n + 1/2}$$

- Plug into Baxter eq. and get recurrence relation for 3 consecutive terms. Solution:

$$\frac{\mu_{12}^{sing}}{Q} = \frac{g^2(Q'(i/2) - Q'(-i/2))}{Q(i/2)}[\psi(1/2 - iu) + \psi(1/2 + iu)]$$

- Using the asymptotics of Euler's $\psi(u) \simeq \log u$ and comparing with the large u asymptotics for μ_{12} we recover standard formula

$$\Delta - L - S = \frac{g^2}{8\pi^2}\,\partial_u \log \frac{Q(u + \frac{i}{2})}{Q(u - \frac{i}{2})}\bigg|_{u=0} + \mathcal{O}(\lambda^2)$$

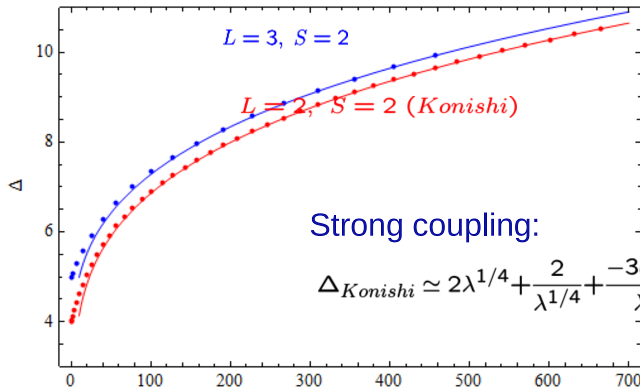
- It follows from P-µ system and not from a pre-defined spin chain Hamiltonian!

Dimensions of twist-2,3,… operators from integrability (QSC)

$\mathrm{Tr}\,(\Phi\,\nabla^S\,\Phi)$

- Numerics, weak and strong coupling from Quantum Spectral Curve;

Gromov,V.K.,Vieira ‘09
Frolov ‘10
Gromov,Valatka ‘12



$\lambda = 16\pi^2 g^2$

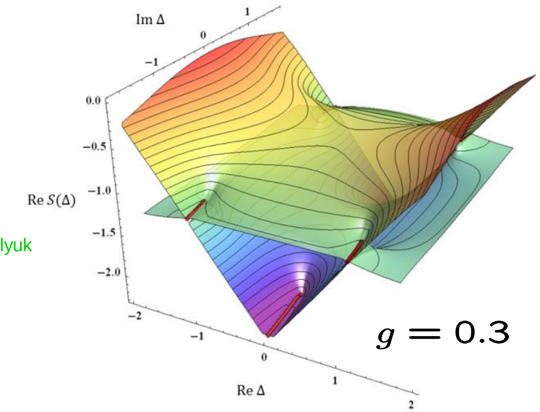
Strong coupling:

$$\Delta_{Konishi} \simeq 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \frac{-3\zeta_3+\frac{1}{2}}{\lambda^{3/4}} + \frac{\frac{15}{2}\zeta_5+6\zeta_3-\frac{1}{2}}{\lambda^{5/4}} + \dots$$

Gromov, Valatka, Sizov, Levkovich-Maslyuk
Gromov,Shenderovich,
Serban, Volin
Roiban, Tseytlin
Vallilo, Mazzucato
Gubser, Klebanov, Polyakov

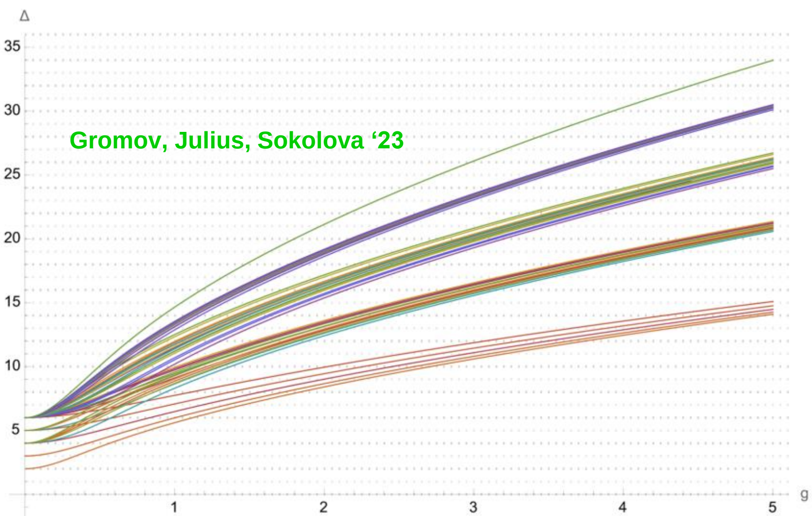
Function of complex conformal spin $\Delta(S,g)$

Gromov, Levkovich-Maslyuk, Sizov ‘15



$g = 0.3$

Recent numerical results with “unlimited” precision (~30-50 digits)



Gromov, Julius, Sokolova ‘23

Weak couping (11 loops)

$$\gamma_{Konishi} = \sum_{j=1}^{\infty} g^{2j}\,\gamma_j$$

Gromov,VK,Leurent, Volin ‘13
Leurent, Serban, Volin ‘12
Volin, Marboe ‘18

$$\begin{aligned}
\gamma_{11} \;=\; & -242508705792 + 107663966208\zeta_3 + 70251466752\zeta_3^2 - 12468142080\zeta_3^3 \\
& +1463132160\zeta_3^4 - 71663616\zeta_3^5 + 180173002752\zeta_5 - 16655486976\zeta_3\zeta_5 \\
& -24628230144\zeta_3^2\zeta_5 - 2895575040\zeta_3^3\zeta_5 + 19278176256\zeta_5^2 - 9619845120\zeta_3\zeta_5^2 \\
& +2504494080\zeta_3^2\zeta_5^2 + \tfrac{882108048384}{175}\zeta_5^3 + 45602231040\zeta_7 + 14993482752\zeta_3\zeta_7 \\
& -12034759680\zeta_3^2\zeta_7 + 1406730240\zeta_3^3\zeta_7 + 30605033088\zeta_5\zeta_7 + 21217637376\zeta_3\zeta_5\zeta_7 \\
& -\tfrac{1309941061632}{275}\zeta_5^2\zeta_7 - 13215327552\zeta_7^2 - 4059901440\zeta_3\zeta_7^2 - 69762034944\zeta_9 \\
& +23284599552\zeta_3\zeta_9 - 3631889664\zeta_3^2\zeta_9 - 11032374528\zeta_5\zeta_9 - 6666706944\zeta_3\zeta_5\zeta_9 \\
& -23148129024\zeta_7\zeta_9 - 10024051968\zeta_9^2 - 54555179184\zeta_{11} + \tfrac{10048541184}{5}\zeta_3\zeta_{11} \\
& -726029568\zeta_3^2\zeta_{11} - 8975463552\zeta_5\zeta_{11} - 22529041920\zeta_7\zeta_{11} - \tfrac{14379934222496}{175}\zeta_{13} \\
& +\tfrac{1504385419392}{35}\zeta_3\zeta_{13} - 30324602880\zeta_5\zeta_{13} - \tfrac{151130039581392}{875}\zeta_{15} - 41375093760\zeta_3\zeta_{15} \\
& -\tfrac{196484147423712}{275}\zeta_{17} + 309361358592\zeta_{19} - 1729880064Z_{11}^{(2)} - \tfrac{1620393984}{5}\zeta_3 Z_{11}^{(2)} \\
& -131383296\zeta_5 Z_{11}^{(2)} + \tfrac{138107420928}{175}Z_{13}^{(2)} + \tfrac{3543865344}{35}\zeta_3 Z_{13}^{(2)} - \tfrac{5716780416}{7}Z_{13}^{(3)} \\
& -\tfrac{674832384}{7}\zeta_3 Z_{13}^{(3)} + \tfrac{48227088384}{175}Z_{15}^{(2)} + \tfrac{3581880576}{25}Z_{15}^{(3)} + 754974720Z_{15}^{(4)} \\
& -\tfrac{854924544}{11}Z_{17}^{(2)} + \tfrac{4963244544}{55}Z_{17}^{(3)} + \tfrac{818159616}{275}Z_{17}^{(4)} + \tfrac{175363688448}{1925}Z_{17}^{(5)}\,. \qquad (A.
\end{aligned}$$

Comments

- Full planar spectrum of N=4 SYM at any coupling encoded into Q-system of QSC
- Q-system of QSC is defined by algebraic (grassmanian) Plucker QQ-relations and analytic structure of 4+4 basic Q-functions
- Pμ-system is a convenient reformulation of QSC
- Systematic perturbation theory is possible for conformal dimensions any single trace operators
- QSC is a consequence of thermodynamic Bethe ansatz for string sigma-model: a striking manifestation of AdS/CFT correspondence.
- Many applications of QSC: cusp dimension of susy-Wilson-Maldacena loop, BFKL-Regge limit, quasi-exact numerics, “bootstrability” – computation of structure constants,…

Cavaglia, Fioravanti,Gromov,Tateo ‘14
Cavaglia, Gromov, Stefanski,Torielli ‘21

- Generalizations: QSC for AdS4/CFT3 (ABJM gauge theory), QSC for AdS3/CFT2
- Fishnet CFT: nontrivial limit of gamm-twisted N=4 SYM (in the last lecture)

References:

- N.Gromov, e-Prints: 1305.1939, 1405.4857
- V.K., S.Leurent, D.Volin , 1510.02100
- V.K. , https://arxiv.org/pdf/1802.02160
- N.Gromov, https://arxiv.org/pdf/1708.03648
- More recent papers of N.Gromov et al…

Lecture IV

Fishnet limit of gamma deformed N=4 SYM

and fishnet CFTs

Outline

- Fishnet CFT is a non-unitary (logarithmic) conformal filed theory. Its particular 4D and 3D examples were discovered as special limit of gamma-twisted $\mathcal{N}$=4 super-Yang-Mills theory and of ABJM theory

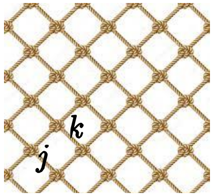
Gurdogan, V.K. '15
Caetano, Gurdogan, V.K. '16

- Integrability of planar $\mathcal{N}$=4 SYM or ABJM becomes in this limit manifest and allows to compute many non-trivial quantities: anomalous dimensions, amplitudes, some structure constants, etc

- Fishnet CFT is dominated by planar graphs with regular lattice structure – fishnets
Such graphs are integrable via the SO(1,D+1) conformal spin chain formalism

Gromov, V.K , Korchemsky, Negro, Sizov '17
V.K., Olivucci 2018

$$\int \prod_i d^D x_i \prod_{<jk>} \frac{1}{|x_j - x_k|^{D/2}}$$



A. Zamolodchikov 1980

- A. Zamolodchikov proposed a general construction of integrable planar graphs, based on Baxter lattice and star-triangle relations

- Many applications….

γ-twisted N=4 SYM and “fishnet” limit

$$\phi_1 = X,\quad \phi_2 = Y,\quad \phi_3 = Z,\quad \psi_1 = \Psi_1,\quad \psi_2 = \Psi_2,\quad \psi_3 = \Psi_3,\quad \lambda = \Psi_4,\quad A$$

- N=4 SYM Lagrangian in SU(3) setting:

Leigh, Strassler
Frolov, Tseytlin
Beisert, Roiban
Lunin, Maldacena

$$\mathcal{L} = N_c \mathrm{tr}\left(F^2 + D\bar{\phi}_i D\phi_i + i\bar{\psi}_j \not{D}\psi_j + i\bar{\lambda}\not{D}\lambda + \right.$$
$$\left. + g^2 [\phi_j,\phi_k]_q \cdot [\bar{\phi}_j,\bar{\phi}_k]_q + i\,g\,\epsilon_{ijk}\bar{\psi}_k[\phi_i,\bar{\psi}_j]_q + g\,\bar{\lambda}[\phi_j,\bar{\psi}_j]_q + \text{ conj.}\right)$$

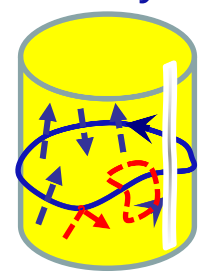
- γ-twist: commutators $\rightarrow$ q-commutators

$$[A\,,B] \rightarrow [A,B]_q \equiv q_{AB}\,A\,B - \frac{1}{q_{AB}}\,B\,A$$ where $$q_{A,B} = e^{-\frac{i}{2}\epsilon^{mjk}\gamma_m\,J_j^A\,J_k^B} = (q_{B,A})^{-1}$$

$J_1^A,\, J_2^A,\, J_3^A \in SO(6)$ - Cartan charges of R-symmetry

$\gamma_1, \gamma_2, \gamma_3$ - twists

- γ-twist -- topological factor on planar graph: quasiperiodic b.c. on cylinder



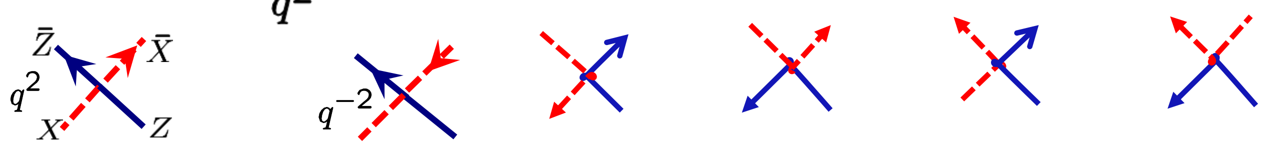
$$\sim\ (q^2)^{\#_\uparrow - \#_\downarrow}$$

typical 4-scalar term

$$q = e^{-i\gamma_3/2}$$

$$\frac{g^2}{16}\mathrm{tr}\left([\Phi^{AB},\Phi^{CD}]_q[\bar{\Phi}^{AB},\bar{\Phi}^{CD}]_q\right) =$$

$$= g^2\mathrm{tr}\left(2q^2\,ZX\bar{Z}\bar{X} + 2\frac{1}{q^2}\,Z\bar{X}\bar{Z}X - Z\bar{X}X\bar{Z} - Z\bar{Z}X\bar{X} - ZX\bar{X}\bar{Z} - Z\bar{Z}\bar{X}X\right) + \dots$$



- Breaks R-symmetry and all supersymmetry: PSU(2,2|4)→SU(2,2)×U(1)^3
Still integrable in planar limit!

Fishnet limit:Chiral CFT and Dynamical “Fishnet”

- Double scaling “fishnet” limit: Strong imaginary twist, weak coupling:

$$g \to 0, \qquad \gamma \to i\infty, \qquad \xi_j = g\, e^{-i\gamma_j/2} - \text{fixed}, \qquad (j = 1,2,3.)$$

Gurdogan, V.K. ‘15

- Chiral CFT from double-scaled γ-twisted N=4 SYM:

$$\mathcal{L} = N_c \mathrm{tr}[-\frac{1}{2}\partial^\mu \bar\phi_i \partial_\mu \phi^i + i\bar\psi^{\dot\alpha}_A \partial^\alpha_{\dot\alpha}\psi^A_\alpha] + \mathcal{L}_{\text{int}}$$

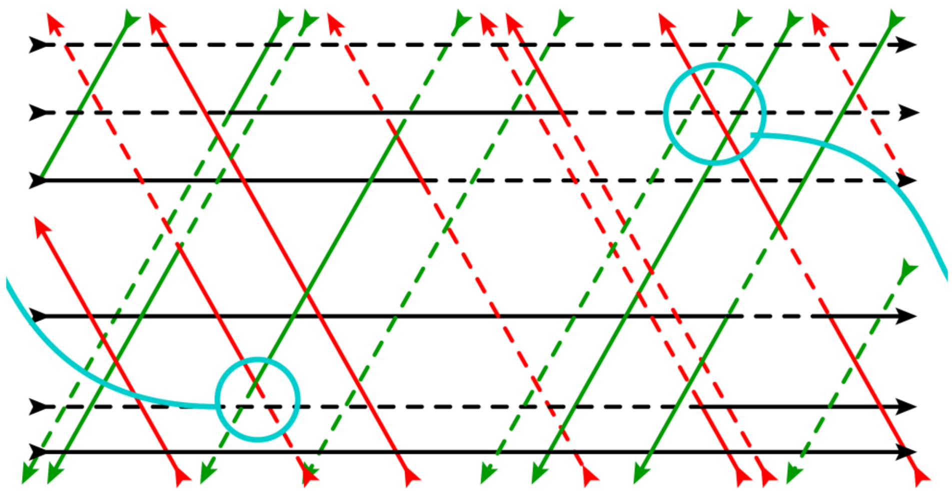
3 flavors of bosons and fermions

$$\mathcal{L}_{\text{int}} = N_c\, \mathrm{tr}[\xi_1^2\, \bar\phi_2\bar\phi_3\phi_2\phi_3 + \xi_2^2\, \bar\phi_3\bar\phi_1\phi_3\phi_1 + \xi_3^2\, \bar\phi_1\bar\phi_2\phi_1\phi_2 +$$

$$+ i\sqrt{\xi_2\xi_3}(\psi^3\phi^1\psi^2 + \bar\psi_3\bar\phi_1\bar\psi_2) + i\sqrt{\xi_1\xi_3}(\psi^1\phi^2\psi^3 + \bar\psi_1\bar\phi_2\bar\psi_3) + i\sqrt{\xi_1\xi_2}(\psi^2\phi^3\psi^1 + \bar\psi_2\bar\phi_3\bar\psi_1)\,].$$

- Feynman graphs form a dynamical fishnet: 3 systems of parallel lines, quartic vertices; solid lines – bosons, dotted lines - fermions

V.K., Olivucci, Preti ‘18



Intersection with fermionic lines should be disentangled into two Yukawa vertices

A challenge: to uncover the underlying integrable spin chain. Some progress…
A step towards understanding of full N=4 SYM integrability.

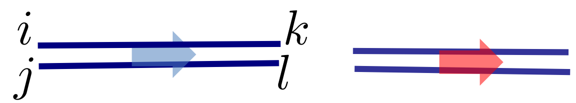
Kade, Staudacher ‘24

Gurdogan, V.K. 2015

Special case: bi-scalar Fishnet CFT$_4$

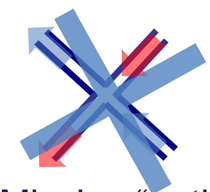
- Take in the full dynakical fishnet CFT $\qquad \xi_1 = \xi_2 = 0, \qquad \xi_3 \equiv \xi$

$$\mathcal{L}[\phi_1,\phi_2] = \frac{N_c}{2}\mathrm{tr}\,\left(\partial^\mu\bar\phi_1\partial_\mu\phi_1 + \partial^\mu\bar\phi_2\partial_\mu\phi_2 + 2\xi^2\,\bar\phi_1\bar\phi_2\phi_1\phi_2\right)$$



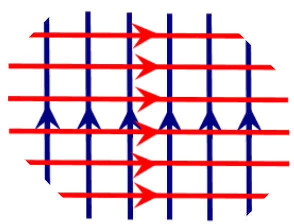
propagators $= \dfrac{\delta_{ik}\,\delta_{jl}}{N_c\,(x-y)^2}$

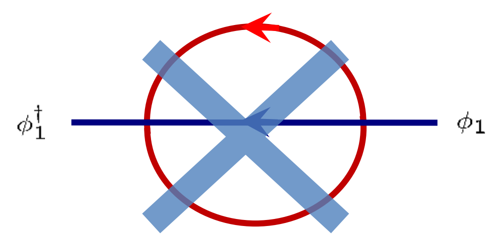
vertex

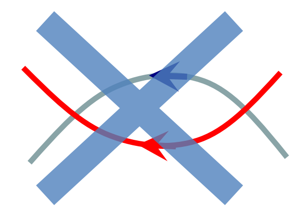


Missing "anti-chiral" vertex

- $\mathcal{N}$= 4 SYM planar graphs reduce, in the bulk, to (very few!) fishnet graphs







- No mass or vertex renormalization! Coupling ξ does not run!

- But there are double-trace counterterms with ξ dependent couplings:

$$\mathrm{tr}\left(\bar\phi_1\bar\phi_2\right)\,\mathrm{tr}\left(\bar\phi_1\phi_2\right), \qquad \mathrm{tr}\left(\phi_1\bar\phi_2\right)\,\mathrm{tr}\left(\bar\phi_1\phi_2\right)), \qquad \mathrm{tr}\left(\phi_j\bar\phi_j\right)\,\mathrm{tr}\left(\phi_j\bar\phi_j\right)$$

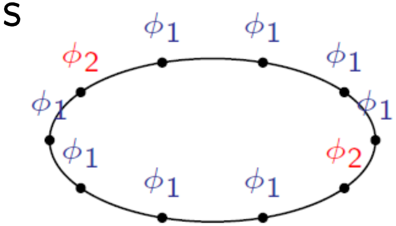
Sieg, Wilhelm '16
Grabner, Gromov, V.K.,Korchemsky, '1

Gurdogan, V.K. 2015

Fishnet planar graphs

- Local operators

$$\mathcal{O}(x) = C^{\mu_1\cdots\mu_n}\,\mathrm{tr}\left[\partial_{\mu_1}\ldots\partial_{\mu_n}(\phi_1)^L(\phi_2)^M(\bar{\phi}_1)^K(\bar{\phi}_2)^N\right](x) + \text{permutations}$$



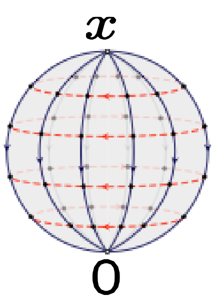
anomalous dimension

- Correlators $\langle\mathcal{O}(x)\ \mathcal{O}(0)\rangle \sim |x|^{-2\Delta_0-2\gamma(\xi)}$

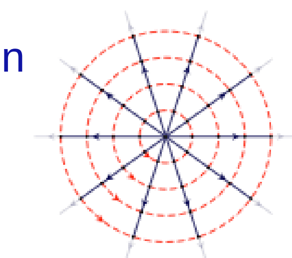
- Examples of 2-point correlators and related conformal dimensions

$\mathrm{tr}[\phi_1(x)]^L$

“vacuum” operator



UV- reduction



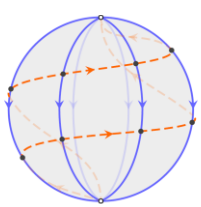
“M-wheel” graphs - divergent!

Need $\epsilon = 4 - D$ regularization

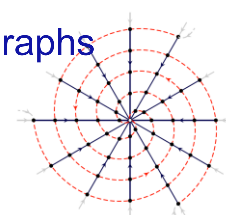
$$= \xi^{2LM}\left(\frac{\#}{\epsilon^M} + \frac{\#}{\epsilon^{M-1}} + \ldots + \frac{C_M}{\epsilon} + \text{finite}\right)$$

$$\gamma(\xi) = \sum_{M=1}^{\infty} C_M\,\xi^{2LM}$$

Multi-magnon spiral graphs



“spiderweb” graphs

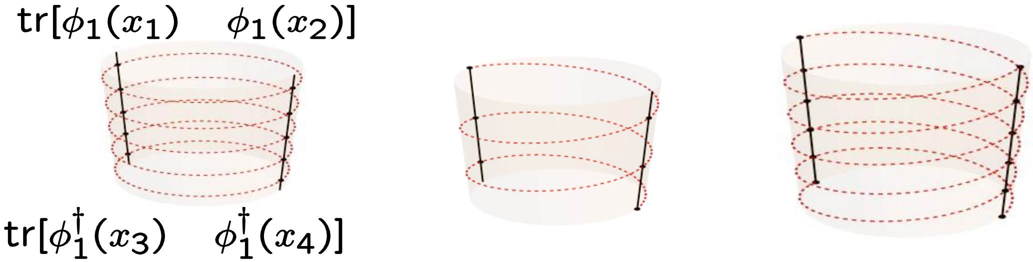


Gurdogan, V.K. 2015
Caetano, Gurdogan, V.K. 2016
Basso, Zhong ’19
Basso, Ferrando, V.K., Zhong ’1

Typical “fishnet” structures in the bulk of graphs. Integrable!

Operators, correlators, graphs…

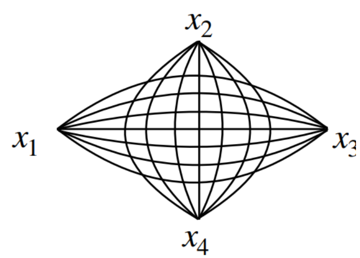
- Explicit computations of L=2 4-point functions



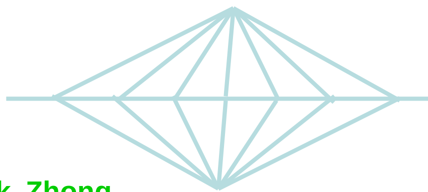
Grabner, Gromov, V.K , Korchemsky '17
V.K., Olivucci 2018
Gromov, V.K , Korchemsky '18
V.K., Olivucci, Preti, '19
Gromov, Sever '20

- Basso-Dixon 4-point functions: expressed through determinant of "ladder" graphs (using Sklyanin SoV)

$$G_{m,n}(x_1, x_2, x_3, x_4) =$$

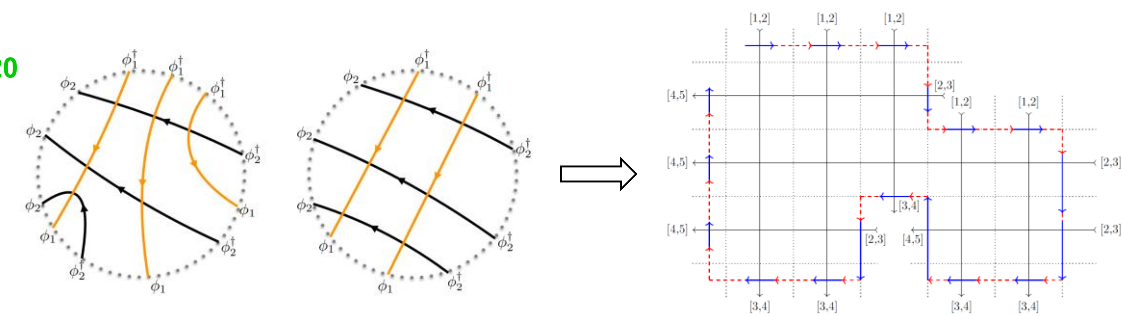


Davidichev, Ushuikina
Basso, Dixon
Derkachev, V.K., Olivucci
Derkachev, Ferrando, Olivucci
Dercachov, Olivucci
Basso, Dixon, Kosover, Kraijenbrink, Zhong
Kostov '23
…



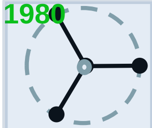
- Amplitudes, Yangian symmetry, Calabi-Yau periods…

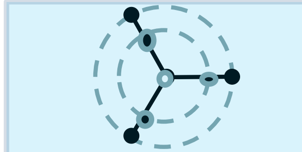
Chicherin, V.K., Mueller, Loebbert, Zheng '17
Corcoran, Loebbert, Miczajka, Muller, Munkler '20
Duhr, Klemm Loebbert, Nega, Porkert'22
V.K., Levkovich-Maslyuk, Mishnyakov '23



Dimension of $\mathtt{tr}(\phi_1)^3$ and periods of wheel graphs from QSC

Broadherst 1980



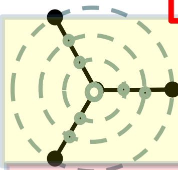


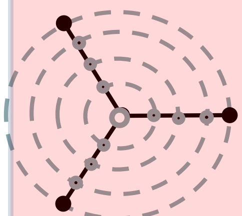
Ahn, Bajnok, Bombardelli, Nepomechie 2013

E.Panzer, 2015

Gurdogan, V.K. '15 (any number of spokes)

In terms of Riemann (multi)-zeta numbers





$$
\begin{aligned}
\boxed{\Delta - 3 =} \; & -12\xi^6\zeta(3) + \xi^{12}\left(189\zeta(7) - 144\zeta(3)^2\right) \\
& +\xi^{18}\Big(-1944\zeta(8,2,1) - 3024\zeta(3)^3 - 3024\zeta(5)\zeta(3)^2 + \frac{198\pi^8\zeta(3)}{175} + 6804\zeta(7)\zeta(3) \\
& +\frac{612\pi^6\zeta(5)}{35} + 270\pi^4\zeta(7) + 5994\pi^2\zeta(9) - \frac{925911\zeta(11)}{8}\Big) + \\
& \xi^{24}\Big(\frac{10368}{5}\pi^4\zeta(8,2,1) + 5184\pi^2\zeta(9,3,1) + 51840\pi^2\zeta(10,2,1) - 148716\zeta(11,3,1) \\
& -1061910\zeta(12,2,1) + 62208\zeta(10,2,1,1,1) - 93312\zeta(3)\zeta(8,2,1) - 288\zeta(3)^5 \\
& +72\gamma\pi^2\zeta(3)^4 - 77760\zeta(3)^4 - \frac{80756\pi^6\zeta(3)^3}{945} - 145152\zeta(5)\zeta(3)^3 - \frac{29}{270}\gamma\pi^8\zeta(3)^2 \\
& +\frac{9504\pi^8\zeta(3)^2}{175} - 879\pi^4\zeta(5)\zeta(3)^2 - 2025\pi^2\zeta(7)\zeta(3)^2 + 244944\zeta(7)\zeta(3)^2 \\
& +186588\zeta(9)\zeta(3)^2 + \frac{2910394\pi^{12}\zeta(3)}{2627625} - 2592\pi^2\zeta(5)^2\zeta(3) + \frac{29376}{35}\pi^6\zeta(5)\zeta(3) \\
& +12960\pi^4\zeta(7)\zeta(3) + 298404\zeta(5)\zeta(7)\zeta(3) + 287712\pi^2\zeta(9)\zeta(3) \\
& -5555466\zeta(11)\zeta(3) + 57672\zeta(5)^3 - 71442\zeta(7)^2 + \frac{13953\pi^{10}\zeta(5)}{1925} + \frac{7293\pi^8\zeta(7)}{175} - \frac{19959\pi^6\zeta(9)}{5} \\
& +\frac{119979\pi^4\zeta(11)}{2} + \frac{10738413\pi^2\zeta(13)}{2} - \frac{4607294013\zeta(15)}{80}\Big) + O\left(\xi^{25}\right)
\end{aligned}
$$

Gromov, V.K , Korchemsky, Negro, Sizov '17

- Based on Quantum Spectral curve of AdS5/CFT5

Gromov, V.K , Leurent, Voilin '13, '14
V.K , Leurent, Voilin '15

Baxter eq.: $$\left(\frac{(\Delta-1)(\Delta-3)}{4u^2} - \frac{i\,\xi^3}{u^3} - 2\right) q(u) + q(u+i) + q(u-i) = 0$$

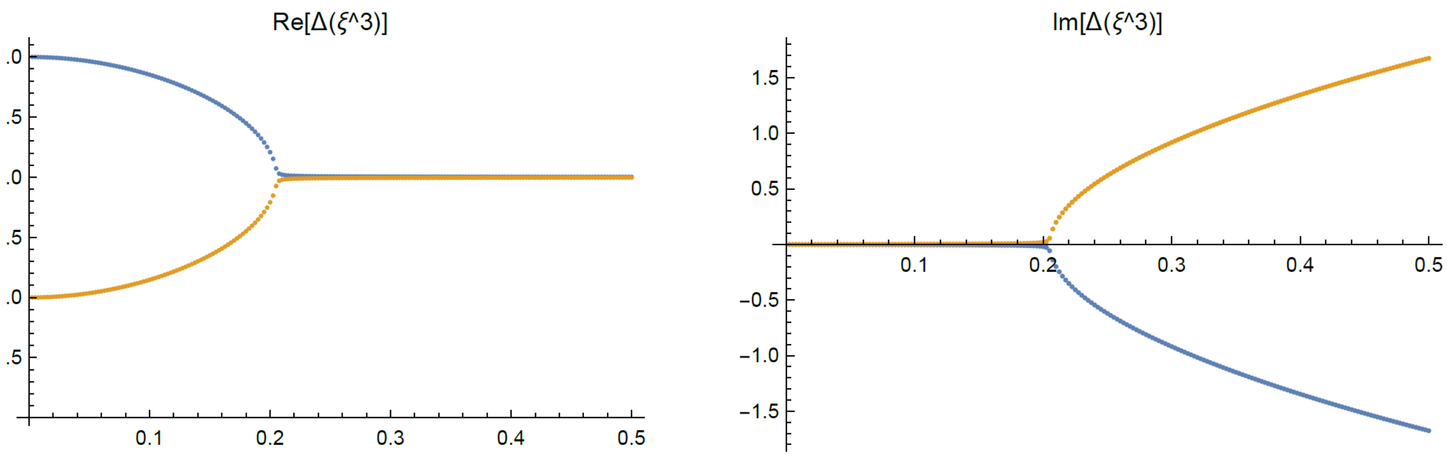
Asymptotics:
$$q_1(u,\xi) \sim u^{\Delta/2-1/2}\left(1 + \frac{\alpha_1}{u} + \frac{\alpha_2}{u^2} + \cdots\right)$$
$$q_2(u,\xi) \sim u^{-\Delta/2+3/2}\left(1 + \frac{\beta_1}{u} + \frac{\beta_2}{u^2} + \cdots\right)$$

Quantization condition $$q_1(0,\xi)\, q_2(0,-\xi) \,+\, q_2(0,-\xi)\, q_1(0,\xi) = 0$$

Interesting relation to Galois coaction on Feynman periods Gurdogan'21

High precision numerics from QSC for dimensions of $\mathrm{tr}(\phi_1)^3$

Gromov, V.K , Korchemsky, Negro, Sizov
(2017)



- Notice that the two dimensions are real for $\xi < \xi_c$,
  but they turn to complex conjugates for $\xi > \xi_c$,
  in accordance with “PT” symmetry of Fishnet CFT

- Generalization to any number of spokes&magnons possible

V.K., Olovucci '22

"PT"-invariance and reality of spectrum

T-transformation (non-unitarity!)

$$\mathcal{L}(\phi_1,\phi_2) \quad \to \quad \overline{\mathcal{L}(\phi_1,\phi_2)}$$

"P"-transformation (transpose)

$$\phi_1 \quad \to \quad \phi_1^t\,, \qquad \phi_2 \quad \to \quad \phi_2^t$$

"PT"-transformation leaves the action invariant:

$$\mathrm{tr}(\phi_1\phi_2\bar\phi_1\bar\phi_2) \quad \overset{T}{\to} \quad \mathrm{tr}(\phi_2\phi_1\bar\phi_2\bar\phi_1) \quad \overset{\text{“}P\text{”}}{\to} \quad \mathrm{tr}(\phi_1\phi_2\bar\phi_1\bar\phi_2)$$

Operators get in general transformed, e.g.

$$\mathrm{tr}(\phi_1\phi_1\phi_2\bar\phi_1) \quad \overset{PT}{\to} \quad \mathrm{tr}(\bar\phi_1\bar\phi_1\bar\phi_2\phi_1)$$

Conformal dimension gets complex conjugate (non-unitary theory!):

$$\left[\langle\bar{\mathcal{O}}(x)\mathcal{O}(0)\rangle\right]^{\mathrm{PT}} = \langle\bar{\mathcal{O}}^{\mathrm{PT}}(x)\mathcal{O}^{\mathrm{PT}}(0)\rangle = |x|^{-2\Delta^*}$$

The spectrum consists of real dimensions or of complex conjugate pairs!

Similar to energy spectrum of non-unitary PT-invariant quantum mechanics

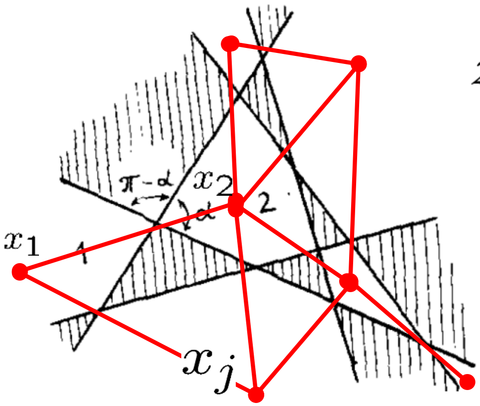
Bender & Boettcher '98

$$\mathcal{H} = \hat{p}^2/2 + x^2\,(ix)^{\epsilon}$$

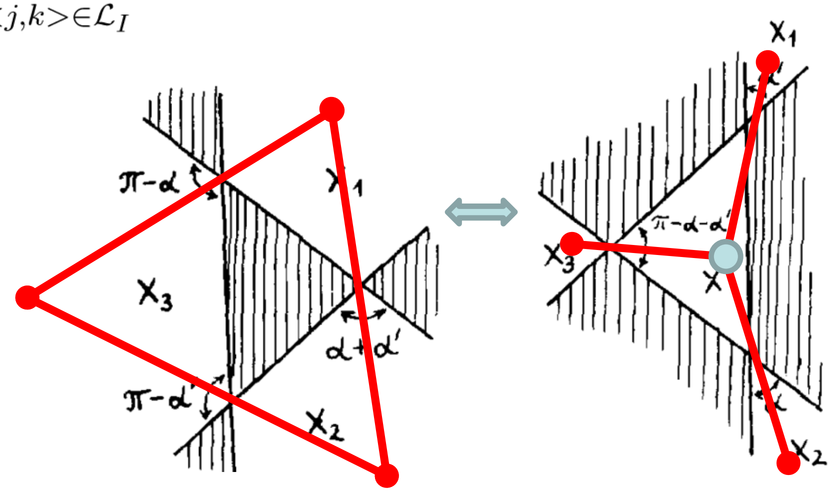
A. Zamolodchikov "Fishnet" graph Integrability

- Feynman graph is dual to Baxter lattice (intersecting straight lines on the plane)
- Dash all the faces connected through the common vertices forming sublattice type I, leaving blank similar complimentary sub-lattice of type II.
- Neighboring vertices connected by propagators
- D-dimensional integration variable x_j in the middle of each blank face

$$\boxed{G_D(x_j,x_k,\alpha_{jk}) = |x_j - x_k|^{\frac{D}{\pi}(\alpha_{jk}-\pi)}}$$

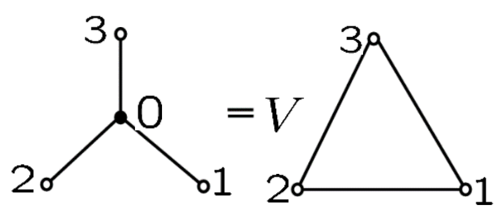


$$\mathcal{Z}_B = \int \prod_{m\in\mathcal{L}_I} d^D x_m \prod_{<j,k>\in\mathcal{L}_I} G_D(x_j,x_k,\alpha_{jk})$$



- We can move a line past intersection due to star-triangle relation (a version of Yang-Baxter relation):

$$\int \frac{d^D x_0}{|x_{10}|^{2a}|x_{20}|^{2b}|x_{30}|^{2c}} = \frac{V(a,b,c)}{|x_{12}|^{D-2c}|x_{23}|^{D-2a}|x_{31}|^{D-2b}}, \qquad (a+b+c=D\,, \quad x_{ij}:=x_i-x_j)$$



if we choose angles as $a=\frac{D}{\pi}\alpha\,,\quad b=\frac{D}{\pi}\beta\,,\quad c=\frac{D}{\pi}\gamma$

$$V(a,b,c) = \pi^{D/2}\frac{\Gamma(\frac{D}{2}-a)\Gamma(\frac{D}{2}-b)\Gamma(\frac{D}{2}-c)}{\Gamma(a)\Gamma(a)\Gamma(c)}$$

Grabner, Gromov, V.K., Korchemsky
V.K., Olivucci

4-point correlator and exact OPE data

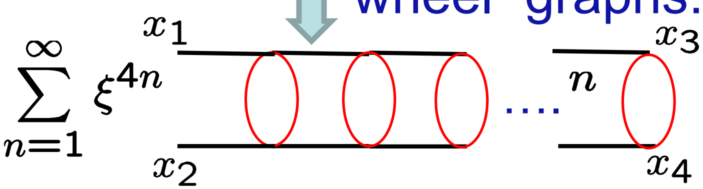
cross-ratios
$u = x_{12}^2 x_{34}^2/(x_{13}^2 x_{24}^2)$
$v = x_{14}^2 x_{23}^2/(x_{13}^2 x_{24}^2)$

- Exact all-loop 4-point correlator

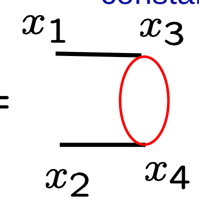
$$\langle \mathrm{tr}[\phi_1(x_1)\phi_1(x_2)]\ \mathrm{tr}[\phi_1^\dagger(x_3)\phi_1^\dagger(x_4)]\rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\frac{D}{2}}} \sum_{\Delta} \sum_{S/2\in\mathbb{Z}_+} C_{\Delta,S}\ u^{(\Delta-S)/2} g_{\Delta,S}(u,v)$$

structure constant — conformal block

“wheel” graphs:

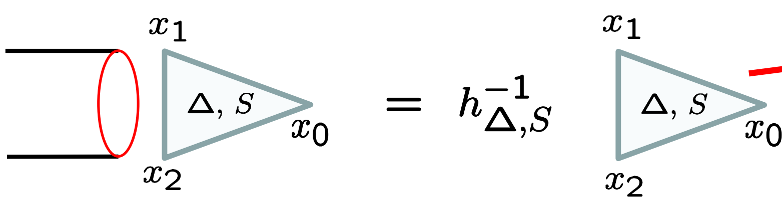


$$= \frac{\xi^4 \mathcal{H}}{1-\xi^4\mathcal{H}} \qquad \text{where} \quad \mathcal{H} =$$



$$= \frac{1}{(x_{13}^2)^{\frac{D}{4}} (x_{24}^2)^{\frac{D}{4}} (x_{34}^2)^{\frac{D}{2}}}$$

- Bethe-Salpeter method & star-triangle: we diagonalize the graph-builing kernel :



$\langle \mathrm{tr}[\phi_1(x_1)\phi_1(x_2)]\ O_{\Delta,S,n}(x_0)\rangle$

conformal 3-point correlator

- Dimensions of “exchange” operators

$$O_{\Delta,S,n}(x_0) = \mathrm{tr}[\phi_1 \partial_+^S \phi_1 (\phi_2^\dagger \phi_2)^n]$$

$$1 - \xi^4/h_{\Delta,S} = 0$$

$$h_{\Delta,S} = \frac{\Gamma\left(\frac{3D}{4} - \frac{\Delta-S}{2}\right)\Gamma\left(\frac{D}{4} + \frac{\Delta+S}{2}\right)}{\Gamma\left(\frac{D}{4} - \frac{\Delta-S}{2}\right)\Gamma\left(-\frac{D}{4} + \frac{\Delta+S}{2}\right)} = \xi^4$$

- Structure constants (D=4):

$$C_{\Delta,S} = \frac{(-1)^{-S}}{(2\pi)^3} \frac{\Gamma(S+2)\Gamma\left(\frac{1}{2}(S+\Delta-1)\right)^2 \Gamma(S-\Delta+4)}{\Gamma(S+1)\Gamma\left(\frac{1}{2}(S-\Delta+5)\right)^2 \Gamma(S+\Delta-1)}$$

- 4-point correlators with 1 and 2 magnons, and a correlator in full chiral theory

Gromov, V.K., Korchemsky — V.K., Olivucci, Preti

Other Results (not full list)

Gurdogan, V.K. ‘16

- Fishnet CFT a unique opportunity to study non-supersymmetric quantum conformal world at any coupling. A window into origins of AdS/CFT integrability.

- Wheel and spiral graphs of any size and systematic solution for spectra of non-compact integrable spin chains. QSC for fisnets

Caetano, Gurdogan, V.K. ’16
Gromov, V.K., Korchemsky, Negro, Sizov ‘15

- Structure constants, 4-point correlation functions, amplitudes…

Basso, Dixon ‘18
Grabner, Gromov, V.K., Korchemsky ‘17
Gromov, Sever ’19
Basso, Caetano, Fleury ‘18
Korchemsky ‘19

- Sklyanin Separated Variables in O(2,D) spin chain

Derkachev, Olivucci ’20
Basso, Ferrando, V.K., Zhong ‘19

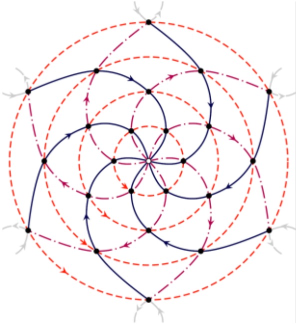
- AdS/CFT?: Fishnet CFT dual to discrete string - “fish-chain”

Gromov, Sever ‘19

- Similar story in ABJM…

Caetano, Gurdogan, V.K. ‘16

ABJM wheel



- Construction of Integrable FCFTs in other dimensions (“Loom” FCFTs)

V.K., Olivucci ‘16, ’23
Alfimov, Ferrando, V.K., Olivucci ‘24

- Yangian symmetry of multi-point FCFT correlators, 2D Calabi-Yau interpretation

Chicherin, V.K., Loebert, Mueller, Zheng ’16,’17
Duhr, Klemm, Loebert, Nega, Porkert ‘23
V.K. Levkovich-Maslyuk, Mishnyakov ’23
Levkovich-Maslyuk, Mishnyakov ’24

Thank you